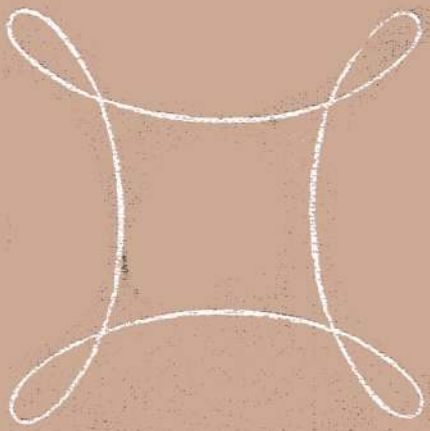
# MATHEMATICAL HANDBOOK

## ELEMENTARY MATHEMATICS

M. VYGODSKY



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ИЗДАТЕЛЬСТВО
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М. Я. ВЫГОДСКИЙ

# СПРАВОЧНИК ПО ЭЛЕМЕНТАРНОЙ МАТЕМАТИКЕ

ИЗДАТЕЛЬСТВО «НАУКА»
МОСКВА

# MATHEMATICAL HANDBOOK

## ELEMENTARY MATHEMATICS

M. VYGODSKY

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GEORGE YANKOVSKY

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# INTRODUCTION

**1. Designation of the handbook.** The purpose of this book is twofold: firstly, it is designed for quick reference to mathematical concepts (to find out what a tangent is, to compute percentages, to recall formulas for the roots of a quadratic equation, etc.). All definitions, rules, formulas, and theorems are supplied with examples. Where required, hints and suggestions are given as to the use of a rule or how to avoid common mistakes, and so forth.

Secondly, the author believes that this handbook can serve as a manual for reviewing the essentials of mathematics and even as a first introductory course in its practical applications.

**2. Handbook and textbook.** An attempt has been made to combine the merits of both books in one text. That this has been successful is evident from numerous letters from the readers, most of whom used it as a textbook. True, this handbook differs radically from an ordinary school textbook, where, especially in the senior classes, the emphasis is placed on reasoning: facts are subordinated to logic. This at any rate is how the student regards the process. In this book, the leading role is played by factual material. This does not in the least mean that the reasoning process is absent. Derivations of formulas are given, but only on occasion, as, for example, when it is necessary to stress the central idea of a given section or to overcome any doubts as to the validity of a result (say, when dealing with operations involving complex numbers). In deciding whether to keep a proof or omit it the author was guided by his own teaching experience.

**3. How to use the handbook.** For quick reference, use the extensive index at the back of the book. If the user has forgotten the exact name of a rule, formula, or mode of solution, he has a detailed table of contents at his disposal in the front.

We strongly advise the user to follow up any additional references he may encounter when investigating a term. Also, much useful information can be gained by reading through

the entire section containing the term or concept he is interested in.

It is wise to pay careful attention to the historical surveys contained in each division. They form an integral part of the book and contribute greatly to a deeper understanding of the subject.

The reader who desires to use this manual as a textbook should pay particular attention to the worked examples. Any proofs that are omitted in the handbook can be filled out by reference to a textbook on the subject either at the same time or later. However, it is well to bear in mind that neither handbook nor textbook alone suffices to give the reader a knowledge of the subject: he must use pencil and paper and work through the examples and problems for himself.

# TABLES

## 1. Some Frequently Encountered Constants

| Quantity | $n$ | $\log_{10} n$ | Quantity | $n$ | $\log_{10} n$ |
|---|---|---|---|---|---|
| $\pi$ | 3.1416 | 0.4971 | | | |
| $2\pi$ | 6.2832 | 0.7982 | $\sqrt[3]{1:\pi}$ | 0.6828 | $\bar{1}.8343$ |
| $3\pi$ | 9.4248 | 0.9743 | $\sqrt[3]{\pi:6}$ | 0.8060 | $\bar{1}.9063$ |
| $4\pi$ | 12.5664 | 1 0992 | | | |
| $4\pi:3$ | 4.1888 | 0 6221 | $\sqrt[3]{3:4\pi}$ | 0.6204 | $\bar{1}.7926$ |
| $\pi:2$ | 1.5708 | 0.1961 | $\sqrt[3]{\pi^2}$ | 2.1450 | 0.3314 |
| $\pi:3$ | 1.0472 | 0.0200 | $e$ | 2.7183 | 0.4343 |
| $\pi:4$ | 0.7854 | $\bar{1}.8951$ | $e^2$ | 7.3891 | 0.8686 |
| $\pi:6$ | 0.5236 | $\bar{1}.7190$ | $\sqrt{e}$ | 1.6487 | 0.2171 |
| $\pi:180$ | 0.0175 | $\bar{2}.2419$ | $\sqrt[3]{e}$ | 1.3956 | 0 1448 |
| $2:\pi$ | 0.6366 | $\bar{1}.8039$ | $1:e$ | 0.3679 | $\bar{1}.5657$ |
| $180:\pi$ | 57.2958 | 1.7581 | $1:e^2$ | 0.1353 | $\bar{1}.1314$ |
| $10800:\pi$ | 3437.7467 | 3.5363 | $\sqrt{1:e}$ | 0.6065 | $\bar{1}.7829$ |
| $648000:\pi$ | 206264.81 | 5.3144 | $\sqrt[3]{1:e}$ | 0.7165 | $\bar{1}.8552$ |
| $1:\pi$ | 0.3183 | $\bar{1}.5029$ | $M=\log e$ | 0.4343 | $\bar{1}.6378$ |
| $1:2\pi$ | 0.1592 | $\bar{1}.2018$ | $\frac{1}{M}=\ln 10$ | 2.3026 | 0.3622 |
| $1:3\pi$ | 0.1061 | $\bar{1}.0257$ | | | |
| $1:4\pi$ | 0.0796 | 2.9008 | 2! | 2 | |
| $\pi^2$ | 9.8696 | 0 9943 | 3! | 6 | |
| $2\pi^2$ | 19.7392 | 1.2953 | 4! | 24 | |
| $\sqrt{\pi}$ | 1.7725 | 0.2486 | 5! | 120 | |
| $\sqrt{2\pi}$ | 2.5066 | 0.3991 | 6! | 720 | |
| $\sqrt{\pi:2}$ | 1.2533 | 0.0981 | 7! | 5040 | |
| $\sqrt{1:\pi}$ | 0.5642 | $\bar{1}.7514$ | 8! | 40,320 | |
| $\sqrt{2:\pi}$ | 0.7979 | $\bar{1}.9019$ | 9! | 362,880 | |
| $\sqrt{3:\pi}$ | 0.9772 | $\bar{1}.9900$ | 10! | 3,628,800 | |
| $\sqrt{4:\pi}$ | 1.1284 | 0.0525 | 11! | 39,916,800 | |
| $\sqrt[3]{\pi}$ | 1.4646 | 0.1657 | 12! | 479,001,600 | |

## 2. Squares, Cubes, Square Roots, Cube Roots, Reciprocals, Circumferences, Areas of Circles, Natural Logarithms

(For three-digit numbers, use interpolation; see Sec. 64. This may result in a slight error in the last digit. On the natural logarithm, ln. see Sec. 128)

| $n$ | $n^2$ | $n^3$ | $\sqrt{n}$ | $\sqrt{10n}$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ | $\frac{1}{n}$ | $\pi n$ | $\frac{\pi n^2}{4}$ | ln $n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1.000 | 3.162 | 1.000 | 2.154 | 4.642 | 1.000 | 3.14 | 0.785 | 0.00000 |
| 2 | 4 | 8 | 1.414 | 4.472 | 1.260 | 2.714 | 5.848 | 0.500 | 6.28 | 3.142 | 0.69315 |
| 3 | 9 | 27 | 1.732 | 5.477 | 1.442 | 3 107 | 6.694 | 0.333 | 9.42 | 7.069 | 1.09861 |
| 4 | 16 | 64 | 2.000 | 6.325 | 1.587 | 3.420 | 7.368 | 0.250 | 12.57 | 12.566 | 1 38629 |
| 5 | 25 | 125 | 2.236 | 7.071 | 1.710 | 3 684 | 7 937 | 0.200 | 15.71 | 19.635 | 1 60944 |
| 6 | 36 | 216 | 2.449 | 7.746 | 1.817 | 3.915 | 8.434 | 0.167 | 18.85 | 28.274 | 1.79176 |
| 7 | 49 | 343 | 2.646 | 8.367 | 1.913 | 4.121 | 8.879 | 0.143 | 21.99 | 38.484 | 1.94591 |
| 8 | 64 | 512 | 2.828 | 8.944 | 2.000 | 4.309 | 9.283 | 0.125 | 25.13 | 50.265 | 2.07944 |
| 9 | 81 | 729 | 3.000 | 9.487 | 2.080 | 4 481 | 9.655 | 0.111 | 28.27 | 63.617 | 2.19722 |
| 10 | 100 | 1000 | 3.162 | 10.000 | 2.154 | 4.642 | 10.000 | 0.100 | 31.42 | 78.540 | 2.30259 |
| 11 | 121 | 1331 | 3.317 | 10.488 | 2.224 | 4.791 | 10.323 | 0.091 | 34.56 | 95.033 | 2.39790 |
| 12 | 144 | 1728 | 3.464 | 10.954 | 2.289 | 4.932 | 10.627 | 0.083 | 37.70 | 113.097 | 2.48491 |
| 13 | 169 | 2197 | 3.606 | 11.402 | 2.351 | 5.066 | 10.914 | 0 077 | 40.84 | 132.73 | 2.56495 |
| 14 | 196 | 2744 | 3.742 | 11.832 | 2.410 | 5.192 | 11.187 | 0.071 | 43.98 | 153.94 | 2.63906 |
| 15 | 225 | 3375 | 3.873 | 12.247 | 2.466 | 5.313 | 11.447 | 0.067 | 47.12 | 176.72 | 2.70805 |
| 16 | 256 | 4096 | 4.000 | 12.649 | 2.520 | 5.429 | 11.696 | 0.062 | 50.27 | 201.06 | 2.77259 |
| 17 | 289 | 4913 | 4.123 | 13.038 | 2.571 | 5.540 | 11.935 | 0.059 | 53.41 | 226.98 | 2.83321 |
| 18 | 324 | 5832 | 4.243 | 13.416 | 2.621 | 5.646 | 12.164 | 0.056 | 56.55 | 254.47 | 2.89037 |
| 19 | 361 | 6859 | 4.359 | 13.784 | 2.668 | 5.749 | 12.386 | 0.053 | 59.69 | 283.53 | 2.94444 |
| 20 | 400 | 8000 | 4.472 | 14.142 | 2.714 | 5.848 | 12.599 | 0.050 | 62.83 | 314.16 | 2.99573 |

*Continued*

| $n$ | $n^4$ | $n^5$ | $\sqrt{n}$ | $\sqrt{10n}$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ | $\frac{1}{n}$ | $\pi n$ | $\frac{\pi n^2}{4}$ | $\ln n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 21 | 441 | 9261 | 4.583 | 14.491 | 2.759 | 5.944 | 12.806 | 0.048 | 65.97 | 346.36 | 3.04452 |
| 22 | 484 | 10,648 | 4.690 | 14.832 | 2.802 | 6.037 | 13.006 | 0.045 | 69.12 | 380.13 | 3.09104 |
| 23 | 529 | 12,167 | 4.796 | 15.166 | 2.844 | 6.127 | 13.200 | 0.043 | 72.26 | 415.48 | 3.13549 |
| 24 | 576 | 13,824 | 4.899 | 15.492 | 2.884 | 6.214 | 13.389 | 0.042 | 75.40 | 452.39 | 3.17805 |
| 25 | 625 | 15,625 | 5.000 | 15.811 | 2.924 | 6.300 | 13.572 | 0.040 | 78.54 | 490.87 | 3.21888 |
| 26 | 676 | 17,576 | 5.099 | 16.125 | 2.962 | 6.383 | 13.751 | 0.038 | 81.68 | 530.93 | 3.25810 |
| 27 | 729 | 19,683 | 5.196 | 16.432 | 3.000 | 6.463 | 13.925 | 0.037 | 84.82 | 572.55 | 3.29584 |
| 28 | 784 | 21,952 | 5.292 | 16.733 | 3.037 | 6.542 | 14.095 | 0.036 | 87.96 | 615.75 | 3.33220 |
| 29 | 841 | 24,389 | 5.385 | 17.029 | 3.072 | 6.619 | 14.260 | 0.034 | 91.11 | 660.52 | 3.36730 |
| 30 | 900 | 27,000 | 5.477 | 17.321 | 3.107 | 6.694 | 14.422 | 0.033 | 94.25 | 706.86 | 3.40120 |
| 31 | 961 | 29,791 | 5.568 | 17.607 | 3.141 | 6.768 | 14.581 | 0.032 | 97.39 | 754.77 | 3.43399 |
| 32 | 1024 | 32,768 | 5.657 | 17.889 | 3.175 | 6.840 | 14.736 | 0.031 | 100.53 | 804.25 | 3.46574 |
| 33 | 1089 | 35,937 | 5.745 | 18.166 | 3.208 | 6.910 | 14.888 | 0.030 | 103.67 | 855.30 | 3.49651 |
| 34 | 1156 | 39,304 | 5.831 | 18.439 | 3.240 | 6.980 | 15.037 | 0.029 | 106.81 | 907.92 | 3.52636 |
| 35 | 1225 | 42,875 | 5.916 | 18.708 | 3.271 | 7.047 | 15.183 | 0.029 | 109.96 | 962.1 | 3.55535 |
| 36 | 1296 | 46,656 | 6.000 | 18.974 | 3.302 | 7.114 | 15.326 | 0.028 | 113.10 | 1017.9 | 3.58352 |
| 37 | 1369 | 50,653 | 6.083 | 19.235 | 3.332 | 7.179 | 15.467 | 0.027 | 116.24 | 1075.2 | 3.61092 |
| 38 | 1444 | 54,872 | 6.164 | 19.494 | 3.362 | 7.243 | 15.605 | 0.026 | 119.4 | 1134.1 | 3.63759 |
| 39 | 1521 | 59,319 | 6.245 | 19.748 | 3.391 | 7.306 | 15.741 | 0.026 | 122.5 | 1194.6 | 3.66356 |
| 40 | 1600 | 64,000 | 6.325 | 20.000 | 3.420 | 7.368 | 15.874 | 0.025 | 125.7 | 1256.6 | 3.68888 |
| 41 | 1681 | 68,921 | 6.403 | 20.248 | 3.448 | 7.429 | 16.005 | 0.024 | 128.8 | 1320.2 | 3.71357 |
| 42 | 1764 | 74,088 | 6.481 | 20.494 | 3.476 | 7.489 | 16.134 | 0.024 | 131.9 | 1385.4 | 3.73767 |
| 43 | 1849 | 79,507 | 6.557 | 20.736 | 3.503 | 7.548 | 16.261 | 0.023 | 135.1 | 1452.2 | 3.76120 |
| 44 | 1936 | 85,184 | 6.633 | 20.976 | 3.530 | 7.606 | 16.386 | 0.023 | 138.2 | 1520.5 | 3.78419 |
| 45 | 2025 | 91,125 | 6.708 | 21.213 | 3.557 | 7.663 | 16.510 | 0.022 | 141.4 | 1590.4 | 3.80666 |

*Continued*

| $n$ | $n^2$ | $n^3$ | $\sqrt{n}$ | $\sqrt{10n}$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ | $\frac{1}{n}$ | $\pi n$ | $\frac{\pi n^2}{4}$ | $\ln n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 46 | 2116 | 97,336 | 6.782 | 21.448 | 3.583 | 7.719 | 16.631 | 0.022 | 144.5 | 1661.9 | 3.82864 |
| 47 | 2209 | 103,823 | 6.856 | 21.679 | 3.609 | 7.775 | 16.751 | 0.021 | 147.7 | 1734.9 | 3.85015 |
| 48 | 2304 | 110,592 | 6.928 | 21.909 | 3.634 | 7.830 | 16.869 | 0.021 | 150.8 | 1809.6 | 3.87120 |
| 49 | 2401 | 117,649 | 7.000 | 22.136 | 3.659 | 7.884 | 16.985 | 0.020 | 153.9 | 1885.7 | 3.89182 |
| 50 | 2500 | 125,000 | 7.071 | 22.361 | 3.684 | 7.937 | 17.100 | 0.020 | 157.1 | 1963.5 | 3.91202 |
| 51 | 2601 | 132,651 | 7.141 | 22.583 | 3.708 | 7.990 | 17.213 | 0.020 | 160.2 | 2042.8 | 3.93183 |
| 52 | 2704 | 140,608 | 7.211 | 22.804 | 3.733 | 8.041 | 17.325 | 0.019 | 163.4 | 2123.7 | 3.95124 |
| 53 | 2809 | 148,877 | 7.280 | 23.022 | 3.756 | 8.093 | 17.435 | 0.019 | 166.4 | 2206.2 | 3.97029 |
| 54 | 2916 | 157,464 | 7.348 | 23.238 | 3.780 | 8.143 | 17.544 | 0.018 | 169.6 | 2290.2 | 3.98898 |
| 55 | 3025 | 166,375 | 7.416 | 23.452 | 3.803 | 8.193 | 17.652 | 0.018 | 172.8 | 2375.8 | 4.00733 |
| 56 | 3136 | 175,616 | 7.483 | 23.664 | 3.826 | 8.243 | 17.758 | 0.018 | 175.9 | 2463.0 | 4.02535 |
| 57 | 3249 | 185,193 | 7.550 | 23.875 | 3.849 | 8.291 | 17.863 | 0.017 | 179.1 | 2551.8 | 4.04306 |
| 58 | 3364 | 195,112 | 7.616 | 24.083 | 3.871 | 8.340 | 17.967 | 0.017 | 182.2 | 2642.1 | 4.06044 |
| 59 | 3481 | 205,379 | 7.681 | 24.290 | 3.893 | 8.387 | 18.070 | 0.017 | 185.4 | 2734.0 | 4.07754 |
| 60 | 3600 | 216,000 | 7.746 | 24.495 | 3.915 | 8.434 | 18.171 | 0.017 | 188.5 | 2827.4 | 4.09434 |
| 61 | 3721 | 226,981 | 7.810 | 24.698 | 3.936 | 8.481 | 18.272 | 0.016 | 191.6 | 2922.5 | 4.11087 |
| 62 | 3844 | 238,328 | 7.874 | 24.900 | 3.958 | 8.527 | 18.371 | 0.016 | 194.8 | 3019.2 | 4.12713 |
| 63 | 3969 | 250,047 | 7.937 | 25.100 | 3.979 | 8.573 | 18.469 | 0.016 | 197.9 | 3117.2 | 4.14313 |
| 64 | 4096 | 262,144 | 8.000 | 25.298 | 4.000 | 8.618 | 18.566 | 0.016 | 201.1 | 3217.0 | 4.15888 |
| 65 | 4225 | 274,625 | 8.062 | 25.495 | 4.021 | 8.662 | 18.663 | 0.015 | 204.2 | 3318.3 | 4.17439 |
| 66 | 4356 | 287,496 | 8.124 | 25.690 | 4.041 | 8.707 | 18.758 | 0.015 | 207.3 | 3421.1 | 4.18965 |
| 67 | 4489 | 300,763 | 8.185 | 25.884 | 4.062 | 8.750 | 18.852 | 0.015 | 210.5 | 3525.6 | 4.20469 |
| 68 | 4624 | 314,432 | 8.246 | 26.077 | 4.082 | 8.794 | 18.945 | 0.015 | 213.6 | 3631.7 | 4.21951 |
| 69 | 4761 | 328,509 | 8.307 | 26.268 | 4.102 | 8.837 | 19.038 | 0.014 | 216.8 | 3739.3 | 4.23411 |
| 70 | 4900 | 343,000 | 8.367 | 26.458 | 4.121 | 8.879 | 19.129 | 0.014 | 219.9 | 3848.4 | 4.24850 |
| 71 | 5041 | 357,911 | 8.426 | 26.646 | 4.141 | 8.921 | 19.220 | 0.014 | 223.1 | 3959.2 | 4.26268 |
| 72 | | [illegible] | 8.485 | 26.833 | 4.160 | 8.963 | 19.310 | 0.014 | 226.2 | 4071.5 | 4.27667 |

*Continued*

| $n$ | $n^2$ | $n^3$ | $\sqrt{n}$ | $\sqrt{10n}$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ | $\frac{1}{n}$ | $\pi n$ | $\frac{\pi n^2}{4}$ | $\ln n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 73 | 5329 | 389.017 | 8.544 | 27.019 | 4.179 | 9.004 | 19.399 | 0.014 | 229.3 | 4185.4 | 4.29046 |
| 74 | 5476 | 405.224 | 8.602 | 27.203 | 4.198 | 9.045 | 19.487 | 0.013 | 232.5 | 4300.8 | 4.30407 |
| 75 | 5625 | 421.875 | 8.660 | 27.386 | 4.217 | 9.086 | 19.574 | 0.013 | 235.6 | 4417.9 | 4.31749 |
| 76 | 5776 | 438.976 | 8.718 | 27.568 | 4.236 | 9.126 | 19.661 | 0.013 | 238.8 | 4536.5 | 4.33073 |
| 77 | 5929 | 456.533 | 8.775 | 27.749 | 4.254 | 9.166 | 19.747 | 0.013 | 241.9 | 4656.6 | 4.34381 |
| 78 | 6084 | 474.552 | 8.832 | 27.928 | 4.273 | 9.205 | 19.832 | 0.013 | 245.0 | 4778.4 | 4.35671 |
| 79 | 6241 | 493.039 | 8.888 | 28.107 | 4.291 | 9.244 | 19.916 | 0.013 | 248.2 | 4901.7 | 4.36945 |
| 80 | 6400 | 512.000 | 8.944 | 28.284 | 4.309 | 9.283 | 20.000 | 0.012 | 251.3 | 5026.6 | 4.38203 |
| 81 | 6561 | 531.441 | 9.000 | 28.460 | 4.327 | 9.322 | 20.083 | 0.012 | 254.5 | 5153.0 | 4.39445 |
| 82 | 6724 | 551.368 | 9.055 | 28.636 | 4.344 | 9.360 | 20.165 | 0.012 | 257.6 | 5281.0 | 4.40672 |
| 83 | 6889 | 571.787 | 9.110 | 28.810 | 4.362 | 9.398 | 20.247 | 0.012 | 260.8 | 5410.6 | 4.41884 |
| 84 | 7056 | 592.704 | 9.165 | 28.983 | 4.380 | 9.435 | 20.328 | 0.012 | 263.9 | 5541.8 | 4.43082 |
| 85 | 7225 | 614.125 | 9.220 | 29.155 | 4.397 | 9.473 | 20.408 | 0.012 | 267.0 | 5674.5 | 4.44265 |
| 86 | 7396 | 636.056 | 9.274 | 29.326 | 4.414 | 9.510 | 20.488 | 0.012 | 270.2 | 5808.8 | 4.45435 |
| 87 | 7569 | 658.503 | 9.327 | 29.496 | 4.431 | 9.546 | 20.567 | 0.011 | 273.3 | 5944.7 | 4.46591 |
| 88 | 7744 | 681.472 | 9.381 | 29.665 | 4.448 | 9.583 | 20.646 | 0.011 | 276.5 | 6082.1 | 4.47734 |
| 89 | 7921 | 704.969 | 9.434 | 29.833 | 4.465 | 9.619 | 20.724 | 0.011 | 279.6 | 6221.1 | 4.48864 |
| 90 | 8100 | 729.000 | 9.487 | 30.000 | 4.481 | 9.655 | 20.801 | 0.011 | 282.7 | 6361.7 | 4.49981 |
| 91 | 8281 | 753.571 | 9.539 | 30.166 | 4.498 | 9.691 | 20.878 | 0.011 | 285.9 | 6503.9 | 4.51086 |
| 92 | 8464 | 778.688 | 9.592 | 30.332 | 4.514 | 9.726 | 20.954 | 0.011 | 289.0 | 6647.6 | 4.52179 |
| 93 | 8649 | 804.357 | 9.644 | 30.496 | 4.531 | 9.761 | 21.029 | 0.011 | 292.2 | 6792.9 | 4.53260 |
| 94 | 8836 | 830.584 | 9.695 | 30.659 | 4.547 | 9.796 | 21.105 | 0.011 | 295.3 | 6939.8 | 4.54329 |
| 95 | 9025 | 857.375 | 9.747 | 30.822 | 4.563 | 9.830 | 21.179 | 0.011 | 298.5 | 7088.2 | 4.55388 |
| 96 | 9216 | 884.736 | 9.798 | 30.984 | 4.579 | 9.865 | 21.253 | 0.010 | 301.6 | 7238.2 | 4.56435 |
| 97 | 9409 | 912.673 | 9.849 | 31.145 | 4.595 | 9.899 | 21.327 | 0.010 | 304.7 | 7389.8 | 4.57471 |
| 98 | 9604 | 941.192 | 9.899 | 31.305 | 4.610 | 9.933 | 21.400 | 0.010 | 307.9 | 7543.0 | 4.58497 |
| 99 | 9801 | 970.299 | 9.950 | 31.464 | 4.626 | 9.967 | 21.472 | 0.010 | 311.0 | 7697.7 | 4.59512 |
| 00 | 10,000 | 1,000.000 | 10.000 | 31.623 | 4.642 | 10.000 | 21.544 | 0.010 | 314.2 | 7854.0 | 4.60517 |

**3. Common Logarithms (see Secs. 131, 132 on how to use the tables: base of natural logarithms: $e = 2.71828$: $\log_{10} e = M = 0.43429$: $\frac{1}{M} = 2.30258$)**

| Mantissas | | | | | | | | | | | Proportional parts | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 2 3 | 4 5 6 | 7 8 9 |
| 10 | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4 9 13<br>4 9 13<br>4 8 12<br>4 8 12 | 17 22 26<br>17 21 25<br>16 21 25<br>16 20 24 | 30 35 39<br>30 34 38<br>29 33 37<br>28 32 36 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4 8 12<br>4 8 11<br>4 7 11 | 16 20 24<br>15 19 23<br>15 18 22 | 27 31 35<br>27 30 34<br>26 29 33 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 | 3 7 11<br>3 7 11<br>3 7 10 | 14 18 21<br>14 17 21<br>14 17 20 | 25 28 32<br>24 28 31<br>24 27 30 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3 7 10<br>3 6 10<br>3 6 9 | 13 17 20<br>13 16 19<br>13 16 19 | 23 27 30<br>23 26 29<br>22 25 28 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3 6 9<br>3 6 9<br>3 6 9 | 13 16 19<br>12 15 18<br>11 14 17 | 22 25 28<br>21 24 27<br>20 23 26 |
| 15 | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3 6 9<br>3 5 8 | 11 14 17<br>11 14 16 | 20 23 26<br>19 22 25 |

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3 5 8<br>3 5 8 | 11 13 16<br>10 13 15 | 19 21 24<br>18 20 23 | | | | | | | | | | | | |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 3 5 8<br>2 5 7 | 10 13 15<br>10 12 15 | 18 20 23<br>17 19 22 | | | | | | | | | | | | |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2 5 7<br>2 5 7 | 9 12 14<br>9 11 13 | 16 19 21<br>16 18 20 | | | | | | | | | | | | |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2 4 7<br>2 4 6 | 9 11 14<br>8 11 13 | 16 18 20<br>15 17 19 | | | | | | | | | | | | |
| 20 | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 4 6<br>2 4 6 | 8 11 13<br>8 10 12 | 15 17 19<br>14 17 19 | | | | | | | | | | | | |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 4 6 | 8 10 12 | 14 16 18 | | | | | | | | | | | | |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 4 6 | 8 10 12 | 14 15 17 | | | | | | | | | | | | |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 4 6 | 7 9 11 | 13 15 17 | | | | | | | | | | | | |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 4 5 | 7 9 11 | 12 14 16 | | | | | | | | | | | | |
| 25 | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 3 5 | 7 9 10 | 12 14 15 | | | | | | | | | | | | |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2 3 5 | 7 8 10 | 11 13 15 | | | | | | | | | | | | |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2 3 5 | 6 8 9 | 11 13 14 | | | | | | | | | | | | |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 3 5 | 6 8 9 | 11 12 14 | | | | | | | | | | | | |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 3 4 | 6 7 9 | 10 12 13 | | | | | | | | | | | | |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 2 3 | 4 5 6 | 7 8 9 | | | | | | | | | | | | |

*Continued*

| Mantissas | | | | | | | | | | | Proportional parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *N* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 30 | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 35 | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 40 | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 43 | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 45 | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 46 | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 50 | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 52 | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 54 | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 55 | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 60 | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 65 | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 70 | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 75 | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |

| Mantissas | | | | | | | | | | | Proportional parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *N* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 80 | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9150 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 85 | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 3 | 3 | 1 | 4 | 5 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 90 | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 91 | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 95 | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| *N* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

## 4. Antilogarithms (see Sec. 133 on how to use the tables)

| Numbers | | | | | | | | | | | Proportional parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *m* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .00 | 1000 | 1002 | 1005 | 1007 | 1009 | 1012 | 1014 | 1016 | 1019 | 1021 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .01 | 1023 | 1026 | 1028 | 1030 | 1033 | 1035 | 1038 | 1040 | 1042 | 1045 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .02 | 1047 | 1050 | 1052 | 1054 | 1057 | 1059 | 1062 | 1064 | 1067 | 1069 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .03 | 1072 | 1074 | 1076 | 1079 | 1081 | 1084 | 1086 | 1089 | 1091 | 1094 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .04 | 1096 | 1099 | 1102 | 1104 | 1107 | 1109 | 1112 | 1114 | 1117 | 1119 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .05 | 1122 | 1125 | 1127 | 1130 | 1132 | 1135 | 1138 | 1140 | 1143 | 1146 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .06 | 1148 | 1151 | 1153 | 1156 | 1159 | 1161 | 1164 | 1167 | 1169 | 1172 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .07 | 1175 | 1178 | 1180 | 1183 | 1186 | 1189 | 1191 | 1194 | 1197 | 1199 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .08 | 1202 | 1205 | 1208 | 1211 | 1213 | 1216 | 1219 | 1222 | 1225 | 1227 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .09 | 1230 | 1233 | 1236 | 1239 | 1242 | 1245 | 1247 | 1250 | 1253 | 1256 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .10 | 1259 | 1262 | 1265 | 1268 | 1271 | 1274 | 1276 | 1279 | 1282 | 1285 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .11 | 1288 | 1291 | 1294 | 1297 | 1300 | 1303 | 1306 | 1309 | 1312 | 1315 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .12 | 1318 | 1321 | 1324 | 1327 | 1330 | 1334 | 1337 | 1340 | 1343 | 1346 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .13 | 1349 | 1352 | 1355 | 1358 | 1361 | 1365 | 1368 | 1371 | 1374 | 1377 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .14 | 1380 | 1384 | 1387 | 1390 | 1393 | 1396 | 1400 | 1403 | 1406 | 1409 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .15 | 1413 | 1416 | 1419 | 1422 | 1426 | 1429 | 1432 | 1435 | 1439 | 1442 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .16 | 1445 | 1449 | 1452 | 1455 | 1459 | 1462 | 1466 | 1469 | 1472 | 1476 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .17 | 1479 | 1483 | 1486 | 1489 | 1493 | 1496 | 1500 | 1503 | 1507 | 1510 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .18 | 1514 | 1517 | 1521 | 1524 | 1528 | 1531 | 1535 | 1538 | 1542 | 1545 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .19 | 1549 | 1552 | 1556 | 1560 | 1563 | 1567 | 1570 | 1574 | 1578 | 1581 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |

*Continued*

| Numbers | | | | | | | | | | | Proportional parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *m* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .20 | 1585 | 1589 | 1592 | 1596 | 1600 | 1603 | 1607 | 1611 | 1614 | 1618 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| .21 | 1622 | 1626 | 1629 | 1633 | 1637 | 1641 | 1644 | 1648 | 1652 | 1656 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| .22 | 1660 | 1663 | 1667 | 1671 | 1675 | 1679 | 1683 | 1687 | 1690 | 1694 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| .23 | 1698 | 1702 | 1706 | 1710 | 1714 | 1718 | 1722 | 1726 | 1730 | 1734 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .24 | 1738 | 1742 | 1746 | 1750 | 1754 | 1758 | 1762 | 1766 | 1770 | 1774 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .25 | 1778 | 1782 | 1786 | 1791 | 1795 | 1799 | 1803 | 1807 | 1811 | 1816 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .26 | 1820 | 1824 | 1828 | 1832 | 1837 | 1841 | 1845 | 1849 | 1854 | 1858 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| .27 | 1862 | 1866 | 1871 | 1875 | 1879 | 1884 | 1888 | 1892 | 1897 | 1901 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| .28 | 1905 | 1910 | 1914 | 1919 | 1923 | 1928 | 1932 | 1936 | 1941 | 1945 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .29 | 1950 | 1954 | 1959 | 1963 | 1968 | 1972 | 1977 | 1982 | 1986 | 1991 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .30 | 1995 | 2000 | 2004 | 2009 | 2014 | 2018 | 2023 | 2028 | 2032 | 2037 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .31 | 2042 | 2046 | 2051 | 2056 | 2061 | 2065 | 2070 | 2075 | 2080 | 2084 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .32 | 2089 | 2094 | 2099 | 2104 | 2109 | 2113 | 2118 | 2123 | 2128 | 2133 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .33 | 2138 | 2143 | 2148 | 2153 | 2158 | 2163 | 2168 | 2173 | 2178 | 2183 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .34 | 2188 | 2193 | 2198 | 2203 | 2208 | 2213 | 2218 | 2223 | 2228 | 2234 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .35 | 2239 | 2244 | 2249 | 2254 | 2259 | 2265 | 2270 | 2275 | 2280 | 2286 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .36 | 2291 | 2296 | 2301 | 2307 | 2312 | 2317 | 2323 | 2328 | 2333 | 2339 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .37 | 2344 | 2350 | 2355 | 2360 | 2366 | 2371 | 2377 | 2382 | 2388 | 2393 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .38 | 2399 | 2404 | 2410 | 2415 | 2421 | 2427 | 2432 | 2438 | 2443 | 2449 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .39 | 2455 | 2460 | 2466 | 2472 | 2477 | 2483 | 2489 | 2495 | 2500 | 2506 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| .40 | 2512 | 2518 | 2523 | 2529 | 2535 | 2541 | 2547 | 2553 | 2559 | 2564 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| .41 | 2570 | 2576 | 2582 | 2588 | 2594 | 2600 | 2606 | 2612 | 2618 | 2624 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| .42 | 2630 | 2636 | 2642 | 2649 | 2655 | 2661 | 2667 | 2673 | 2679 | 2685 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| .43 | 2692 | 2693 | 2704 | 2710 | 2716 | 2723 | 2729 | 2735 | 2742 | 2748 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| .44 | 2754 | 2761 | 2767 | 2773 | 2780 | 2786 | 2793 | 2799 | 2805 | 2812 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| .45 | 2818 | 2825 | 2831 | 2838 | 2844 | 2851 | 2858 | 2864 | 2871 | 2877 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .46 | 2884 | 2891 | 2897 | 2904 | 2911 | 2917 | 2927 | 2931 | 2938 | 2944 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .47 | 2951 | 2958 | 2965 | 2972 | 2979 | 2985 | 2992 | 2999 | 3006 | 3013 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .48 | 3020 | 3027 | 3034 | 3041 | 3048 | 3055 | 3062 | 3069 | 3076 | 3083 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| .49 | 3090 | 3097 | 3105 | 3112 | 3119 | 3126 | 3133 | 3141 | 3148 | 3155 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| .50 | 3162 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| .51 | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| .52 | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| .53 | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| .54 | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| .55 | 3548 | 3556 | 3565 | 3573 | 3581 | 3589 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| .56 | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 2698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| .57 | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| .58 | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| .59 | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| *m* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

*Continued*

| Numbers | | | | | | | | | | | Proportional parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *m* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .60 | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| .61 | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .62 | 4169 | 4278 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .63 | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .64 | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .65 | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .66 | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| .67 | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| .68 | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| .69 | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| .70 | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| .71 | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| .72 | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| .73 | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| .74 | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| .75 | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| .76 | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5848 | 5861 | 5875 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| .77 | 5888 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5984 | 5998 | 6012 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| .78 | 6026 | 6039 | 6053 | 6067 | 6081 | 6095 | 6109 | 6124 | 6138 | 6152 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| .79 | 6166 | 6180 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| .80 | 6310 | 6324 | 6339 | 6353 | 6368 | 6383 | 6397 | 6412 | 6427 | 6442 | 1 3 4 | 6 7 9 | 10 12 13 |
| .81 | 6457 | 6471 | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 2 3 5 | 6 8 9 | 11 12 14 |
| .82 | 6607 | 6622 | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 2 3 5 | 6 8 9 | 11 12 14 |
| .83 | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 2 3 5 | 6 8 9 | 11 13 14 |
| .84 | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 2 3 5 | 6 8 10 | 11 13 15 |
| .85 | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7228 | 2 3 5 | 7 8 10 | 12 13 15 |
| .86 | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 2 3 5 | 7 8 10 | 12 13 15 |
| .87 | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 2 3 5 | 7 9 10 | 12 14 16 |
| .88 | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 2 4 5 | 7 9 11 | 12 14 16 |
| .89 | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 2 4 5 | 7 9 11 | 13 14 16 |
| .90 | 7943 | 7962 | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 2 4 6 | 7 9 11 | 13 15 17 |
| .91 | 8128 | 8147 | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 2 4 6 | 8 9 11 | 13 15 17 |
| .92 | 8318 | 8337 | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 2 4 6 | 8 10 12 | 14 15 17 |
| .93 | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 2 4 6 | 8 10 12 | 14 16 18 |
| .94 | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 2 4 6 | 8 10 12 | 14 16 18 |
| .95 | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 4 6 | 8 10 12 | 15 17 19 |
| .96 | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 4 6 | 8 11 13 | 15 17 19 |
| .97 | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 4 7 | 9 11 13 | 15 17 20 |
| .98 | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 4 7 | 9 11 13 | 16 18 20 |
| .99 | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 5 7 | 9 11 14 | 16 18 20 |

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 2 3 | 4 5 6 | 7 8 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

## 5. Logarithms of Trigonometric Functions

(see Sec. 186—188; in the column headings log sin, log tan, log cos, all characteristics are increased by 10)

| ° | ′ | log sin | d | log tan | cd | log cot | d | log cos | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | −∞ | — | −∞ | — | +∞ | | 10.0000 | 0 | 90 |
| | 10 | 7.4637 | 3011 | 7.4637 | 3011 | 2.5363 | | 9.999998 | 50 | |
| | 20 | 7.7648 | 1760 | 7.7648 | 1761 | 2.2352 | | 9.99999 | 40 | |
| | 30 | 7.9408 | 1250 | 7.9409 | 1249 | 2.0591 | | 9.99998 | 30 | |
| | 40 | 8.0658 | 969 | 8.0658 | 969 | 1.9342 | | 9.99997 | 20 | |
| | 50 | 8.1627 | 792 | 8.1627 | 792 | 1.8373 | | 9.9999 | 10 | |
| 1 | 0 | 8.2419 | 669 | 8.2419 | 670 | 1.7581 | | 9.9999 | 0 | 89 |
| | 10 | 8.3088 | 580 | 8.3089 | 580 | 1.6911 | | 9.9999 | 50 | |
| | 20 | 8.3668 | 511 | 8.3669 | 512 | 1.6331 | | 9.9999 | 40 | |
| | 30 | 8.4179 | 458 | 8.4181 | 457 | 1.5819 | 1 | 9.9999 | 30 | |
| | 40 | 8.4637 | 413 | 8.4638 | 415 | 1.5362 | | 9.9998 | 20 | |
| | 50 | 8.5050 | 378 | 8.5053 | 378 | 1.4947 | 1 | 9.9998 | 10 | |
| 2 | 0 | 8.5428 | 348 | 8.5431 | 348 | 1.4569 | | 9.9997 | 0 | 88 |
| | 10 | 8.5776 | 321 | 8.5779 | 322 | 1.4221 | 1 | 9.9997 | 50 | |
| | 20 | 8.6097 | 300 | 8.6101 | 300 | 1.3899 | | 9.9996 | 40 | |
| | 30 | 8.6397 | 280 | 8.6401 | 281 | 1.3599 | 1 | 9.9996 | 30 | |
| | 40 | 8.6677 | 263 | 8.6682 | 263 | 1.3318 | | 9.9995 | 20 | |
| | 50 | 8.6940 | 248 | 8.6945 | 249 | 1.3055 | 1 | 9.9995 | 10 | |
| 3 | 0 | 8.7188 | 235 | 8.7194 | 235 | 1.2806 | 1 | 9.9994 | 0 | 87 |
| | 10 | 8.7423 | 222 | 8.7429 | 223 | 1.2571 | 1 | 9.9993 | 50 | |
| | 20 | 8.7645 | 212 | 8.7652 | 213 | 1.2348 | 1 | 9.9993 | 40 | |
| | 30 | 8.7857 | 202 | 8.7865 | 202 | 1.2135 | 1 | 9.9992 | 30 | |
| | 40 | 8.8059 | 192 | 8.8067 | 194 | 1.1933 | | 9.9991 | 20 | |
| | 50 | 8.8251 | 185 | 8.8261 | 185 | 1.1739 | 1 | 9.9990 | 10 | |
| 4 | 0 | 8.8436 | 177 | 8.8446 | 178 | 1.1554 | | 9.9989 | 0 | 86 |
| | 10 | 8.8613 | 170 | 8.8624 | 171 | 1.1376 | 1 | 9.9989 | 50 | |
| | 20 | 8.8783 | 163 | 8.8795 | 165 | 1.1205 | 1 | 9.9988 | 40 | |
| | 30 | 8.8946 | 158 | 8.8960 | 158 | 1.1040 | 1 | 9.9987 | 30 | |
| | 40 | 8.9104 | 152 | 8.9118 | 154 | 1.0882 | 1 | 9.9986 | 20 | |
| | 50 | 8.9256 | 147 | 8.9272 | 148 | 1.0728 | 2 | 9.9985 | 10 | |
| | | log cos | d | log cot | cd | log tan | d | log sin. | ′ | ° |

*Continued*

| ° | ′ | log sin | d | log tan | cd | log cot | d | log cos | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0 | 8.9403 | 142 | 8 9420 | 143 | 1 0580 | 1 | 9.9983 | 0 | 85 |
| | 10 | 8.9545 | 137 | 8.9563 | 138 | 1.0437 | 1 | 9.9982 | 50 | |
| | 20 | 8.9682 | 134 | 8 9701 | 135 | 1 0299 | 1 | 9.9981 | 40 | |
| | 30 | 8.9816 | 129 | 8.9836 | 130 | 1 0164 | 1 | 9.9980 | 30 | |
| | 40 | 8.9945 | 125 | 8.9966 | 127 | 1.0034 | 2 | 9.9979 | 20 | |
| | 50 | 9.0070 | 122 | 9.0093 | 123 | 0 9907 | 1 | 9.9977 | 10 | |
| 6 | 0 | 9.0192 | 119 | 9.0216 | 120 | 0.9784 | 1 | 9.9976 | 0 | 84 |
| | 10 | 9.0311 | 115 | 9.0336 | 117 | 0.9664 | 2 | 9.9975 | 50 | |
| | 20 | 9.0426 | 113 | 9.0453 | 114 | 0.9547 | 1 | 9.9973 | 40 | |
| | 30 | 9.0539 | 109 | 9.0467 | 111 | 0.9433 | 1 | 9.9972 | 30 | |
| | 40 | 9.0648 | 107 | 9.0678 | 108 | 0.9322 | 2 | 9.9971 | 20 | |
| | 50 | 9.0755 | 104 | 9.0786 | 105 | 0.9214 | 1 | 9.9969 | 10 | |
| 7 | 0 | 9.0859 | 102 | 9.0891 | 104 | 0.9109 | 2 | 9.9968 | 0 | 83 |
| | 10 | 9.0961 | 99 | 9.0995 | 101 | 0.9005 | 2 | 9.9966 | 50 | |
| | 20 | 9.1060 | 97 | 9.1096 | 98 | 0.8904 | 1 | 9.9964 | 40 | |
| | 30 | 9.1157 | 95 | 9 1194 | 97 | 0.8806 | 2 | 9.9963 | 30 | |
| | 40 | 9.1252 | 93 | 9 1291 | 94 | 0.8709 | 2 | 9.9961 | 20 | |
| | 50 | 9.1345 | 91 | 9.1385 | 93 | 0.8615 | 1 | 9.9959 | 10 | |
| 8 | 0 | 9.1436 | 89 | 9.1478 | 91 | 0.8522 | 2 | 9.9958 | 0 | 82 |
| | 10 | 9.1525 | 87 | 9.1569 | 89 | 0.8431 | 2 | 9.9956 | 50 | |
| | 20 | 9.1612 | 85 | 9 1658 | 87 | 0.8342 | 2 | 9.9954 | 40 | |
| | 30 | 9.1697 | 84 | 9.1745 | 86 | 0.8255 | 2 | 9.9952 | 30 | |
| | 40 | 9.1781 | 82 | 9.1831 | 84 | 0.8169 | 2 | 9.9950 | 20 | |
| | 50 | 9.1863 | 80 | 9.1915 | 82 | 0 8085 | 2 | 9.9948 | 10 | |
| 9 | 0 | 9.1943 | 79 | 9.1997 | 81 | 0 8003 | 2 | 9.9946 | 0 | 81 |
| | 10 | 9.2022 | 78 | 9.2078 | 80 | 0 7922 | 2 | 9.9944 | 50 | |
| | 20 | 9.2100 | 76 | 9.2158 | 78 | 0.7842 | 2 | 9.9942 | 40 | |
| | 30 | 9.2176 | 75 | 9.2236 | 77 | 0.7764 | 2 | 9.9940 | 30 | |
| | 40 | 9.2251 | 73 | 9.2313 | 76 | 0.7687 | 2 | 9.9938 | 20 | |
| | 50 | 9.2324 | 73 | 9.2389 | 74 | 0 7611 | 2 | 9.9936 | 10 | |
| 10 | 0 | 9.2397 | 71 | 9.2463 | 73 | 0.7537 | 3 | 9.9934 | 0 | 80 |
| | 10 | 9.2468 | 70 | 9.2536 | 73 | 0.7464 | 2 | 9.9931 | 50 | |
| | 20 | 9.2538 | 68 | 9.2609 | 71 | 0.7391 | 2 | 9.9929 | 40 | |
| | 30 | 9.2606 | 68 | 9 2680 | 70 | 0.7320 | 3 | 9.9927 | 30 | |
| | 40 | 9.2674 | 66 | 9.2750 | 69 | 0.7250 | 2 | 9.9924 | 20 | |
| | 50 | 9.2740 | 66 | 9.2819 | 68 | 0.7181 | 3 | 9.9922 | 10 | |
| | | log cos | d | log cot | cd | log tan | d | log sin | | ° |

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| ° | ′ | log sin | d | log tan | cd | log cot | d | log cos | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 17 | 0 | 9.4659 | | 9.4853 | | 0.5147 | | 9.9806 | 0 | 73 |
| | 10 | 9.4700 | 41 | 9.4898 | 45 | 0.5102 | 4 | 9.9802 | 50 | |
| | 20 | 9.4741 | 41 | 9.4943 | 45 | 0 5057 | 1 | 9.9798 | 40 | |
| | 30 | 9 4781 | 40 | 9 4987 | 44 | 0.5013 | 4 | 9.9794 | 30 | |
| | 40 | 9.4821 | 40 | 9.5031 | 44 | 0.4969 | 4 | 9.9790 | 20 | |
| | 50 | 9.4861 | 40 | 9.5075 | 44 | 0.4925 | 4 | 9.9786 | 10 | |
| 18 | 0 | 9.4900 | 39 | 9.5118 | 43 | 0 4882 | 4 | 9.9782 | 0 | 72 |
| | 10 | 9.4939 | 39 | 9.5161 | 43 | 0 4839 | 4 | 9.9778 | 50 | |
| | 20 | 9.4977 | 38 | 9.5203 | 42 | 0.4797 | 4 | 9.9774 | 40 | |
| | 30 | 9.5015 | 38 | 9.5245 | 42 | 0.4755 | 5 | 9.9770 | 30 | |
| | 40 | 9.5052 | 37 | 9.5287 | 42 | 0 4713 | 4 | 9.9765 | 20 | |
| | 50 | 9.5090 | 38 | 9.5329 | 42 | 0.4671 | 4 | 9.9761 | 10 | |
| 19 | 0 | 9.5126 | 36 | 9.5370 | 41 | 0.4630 | 5 | 9.9757 | 0 | 71 |
| | 10 | 9.5163 | 37 | 9.5411 | 41 | 0.4589 | 4 | 9.9752 | 50 | |
| | 20 | 9.5199 | 36 | 9.5451 | 40 | 0.4549 | 5 | 9.9748 | 40 | |
| | 30 | 9.5235 | 36 | 9.5491 | 40 | 0.4509 | 4 | 9.9743 | 30 | |
| | 40 | 9.5270 | 35 | 9.5531 | 40 | 0.4469 | 5 | 9.9739 | 20 | |
| | 50 | 9.5306 | 36 | 9.5571 | 40 | 0.4429 | 4 | 9.9734 | 10 | |
| 20 | 0 | 9.5341 | 35 | 9.5611 | 40 | 0.4389 | 5 | 9.9730 | 0 | 70 |
| | 10 | 9.5375 | 34 | 9.5650 | 39 | 0.4350 | 1 | 9.9725 | 50 | |
| | 20 | 9.5409 | 34 | 9.5689 | 39 | 0.4311 | 5 | 9.9721 | 40 | |
| | 30 | 9.5443 | 34 | 9.5727 | 38 | 0.4273 | 5 | 9.9716 | 30 | |
| | 40 | 9.5477 | 34 | 9.5766 | 39 | 0.4234 | 5 | 9.9711 | 20 | |
| | 50 | 9.5510 | 33 | 9.5804 | 38 | 0.4196 | 5 | 9.9706 | 10 | |
| 21 | 0 | 9.5543 | 33 | 9.5842 | 38 | 0.4158 | 4 | 9.9702 | 0 | 69 |
| | 10 | 9.5576 | 33 | 9.5879 | 37 | 0.4121 | 5 | 9.9697 | 50 | |
| | 20 | 9.5609 | 33 | 9.5917 | 38 | 0.4083 | 5 | 9.9692 | 40 | |
| | 30 | 9.5641 | 32 | 9.5954 | 37 | 0 4046 | 5 | 9.9687 | 30 | |
| | 40 | 9.5673 | 32 | 9.5991 | 37 | 0 4009 | 5 | 9.9682 | 20 | |
| | 50 | 9.5704 | 31 | 9.6028 | 37 | 0 3972 | 5 | 9.9677 | 10 | |
| 22 | 0 | 9.5736 | 32 | 9.6064 | 36 | 0 3936 | 5 | 9.9672 | 0 | 68 |
| | 10 | 9.5767 | 31 | 9.6100 | 36 | 0 3900 | 6 | 9.9667 | 50 | |
| | 20 | 9.5798 | 31 | 9.6136 | 36 | 0.3864 | 5 | 9.9661 | 40 | |
| | 30 | 9.5828 | 30 | 9.6172 | 36 | 0.3828 | 5 | 9.9656 | 30 | |
| | 40 | 9.5859 | 31 | 9.6208 | 36 | 0.3792 | 5 | 9.9651 | 20 | |
| | 50 | 9.5889 | 30 | 9.6243 | 35 | 0.3757 | 6 | 9.9646 | 10 | |
| | | | 30 | | 36 | | | | | |
| | | log cos | d | log cot | d | log tan | d | log sin | ′ | ° |

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| ° | ′ | log sin | d | log tan | cd | log cot | d | log cos | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 23 | 0 | 9.5919 | | 9 6279 | | 0 3721 | | 9.9640 | 0 | 67 |
| | 10 | 9.5948 | 29 | 9 6314 | 35 | 0 3686 | 5 | 9.9635 | 50 | |
| | 20 | 9.5978 | 30 | 9.6348 | 34 | 0.3652 | 6 | 9.9629 | 40 | |
| | 30 | 9.6007 | 29 | 9.6383 | 35 | 0 3617 | 5 | 9.9624 | 30 | |
| | 40 | 9.6036 | 29 | 9.6417 | 34 | 0.3583 | 6 | 9.9618 | 20 | |
| | 50 | 9.6065 | 29 | 9.6452 | 35 | 0 3548 | 5 | 9.9613 | 10 | |
| 24 | 0 | 9.6093 | 28 | 9 6486 | 34 | 0.3514 | 6 | 9.9607 | 0 | 66 |
| | 10 | 9.6121 | 28 | 9.6520 | 34 | 0.3480 | 5 | 9.9602 | 50 | |
| | 20 | 9.6149 | 28 | 9 6553 | 33 | 0.3447 | 6 | 9.9596 | 40 | |
| | 30 | 9.6177 | 28 | 9.6587 | 34 | 0.3413 | 6 | 9.9590 | 30 | |
| | 40 | 9.6205 | 28 | 9.6620 | 33 | 0.3380 | 6 | 9.9584 | 20 | |
| | 50 | 9.6232 | 27 | 9.6654 | 34 | 0 3346 | 5 | 9.9579 | 10 | |
| 25 | 0 | 9.6259 | 27 | 9.6687 | 33 | 0.3313 | 6 | 9.9573 | 0 | 65 |
| | 10 | 9.6286 | 27 | 9.6720 | 33 | 0.3280 | 6 | 9.9567 | 50 | |
| | 20 | 9.6313 | 27 | 9 6752 | 32 | 0.3248 | 6 | 9.9561 | 40 | |
| | 30 | 9.6340 | 27 | 9.6785 | 33 | 0.3215 | 6 | 9.9555 | 30 | |
| | 40 | 9.6366 | 26 | 9.6817 | 32 | 0.3183 | 6 | 9.9549 | 20 | |
| | 50 | 9.6392 | 26 | 9 6850 | 33 | 0.3150 | 6 | 9.9543 | 10 | |
| 26 | 0 | 9.6418 | 26 | 9.6882 | 32 | 0.3118 | 6 | 9.9537 | 0 | 64 |
| | 10 | 9.6444 | 26 | 9.6914 | 32 | 0.3086 | 7 | 9.9530 | 50 | |
| | 20 | 9.6470 | 26 | 9.6946 | 32 | 0.3054 | 6 | 9.9524 | 40 | |
| | 30 | 9.6495 | 25 | 9.6977 | 31 | 0.3023 | 6 | 9.9518 | 30 | |
| | 40 | 9.6521 | 26 | 9.7009 | 32 | 0.2991 | 6 | 9.9512 | 20 | |
| | 50 | 9.6546 | 25 | 9.7040 | 31 | 0.2960 | 7 | 9.9505 | 10 | |
| 27 | .0 | 9.6570 | 24 | 9.7072 | 32 | 0.2928 | 6 | 9.9499 | 0 | 63 |
| | 10 | 9.6595 | 25 | 9.7103 | 31 | 0.2897 | 7 | 9.9492 | 50 | |
| | 20 | 9.6620 | 25 | 9.7134 | 31 | 0.2866 | 6 | 9.9486 | 40 | |
| | 30 | 9.6644 | 24 | 9.7165 | 31 | 0.2835 | 7 | 9.9479 | 30 | |
| | 40 | 9.6668 | 24 | 9.7196 | 31 | 0.2804 | 6 | 9.9473 | 20 | |
| | 50 | 9.6692 | 24 | 9.7226 | 30 | 0.2774 | 7 | 9.9466 | 10 | |
| 28 | 0 | 9.6716 | 24 | 9.7257 | 31 | 0.2743 | 7 | 9.9459 | 0 | 62 |
| | 10 | 9.6740 | 24 | 9.7287 | 30 | 0.2713 | 6 | 9.9453 | 50 | |
| | 20 | 9.6763 | 23 | 9.7317 | 30 | 0.2683 | 7 | 9.9446 | 40 | |
| | 30 | 9.6787 | 24 | 9.7348 | 31 | 0.2652 | 7 | 9.9439 | 30 | |
| | 40 | 9.6810 | 23 | 9.7378 | 30 | 0.2622 | 7 | 9.9432 | 20 | |
| | 50 | 9.6833 | 23 | 9.7408 | 30 | 0.2592 | 7 | 9.9425 | 10 | |
| | | log cos | d | log cot | cd | log tan | d | log sin | ′ | ° |

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| ° | ′ | log sin | d | log tan | cd | log cot | d | log cos | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 29 | 0 | 9.6856 | | 9.7438 | | 0.2562 | | 9.9418 | 0 | 61 |
| | 10 | 9.6878 | 22 | 9.7467 | 29 | 0.2533 | 7 | 9.9411 | 50 | |
| | 20 | 9.6901 | 23 | 9.7497 | 30 | 0.2503 | 7 | 9.9404 | 40 | |
| | 30 | 9.6923 | 22 | 9.7526 | 29 | 0.2474 | 7 | 9.9397 | 30 | |
| | 40 | 9.6946 | 23 | 9.7556 | 30 | 0.2444 | 7 | 9.9390 | 20 | |
| | 50 | 9.6968 | 22 | 9.7585 | 29 | 0.2415 | 7 | 9.9383 | 10 | |
| 30 | 0 | 9.6990 | 22 | 9.7614 | 29 | 0.2386 | 8 | 9.9375 | 0 | 60 |
| | 10 | 9.7012 | 22 | 9.7644 | 30 | 0.2356 | 7 | 9.9368 | 50 | |
| | 20 | 9.7033 | 21 | 9.7673 | 29 | 0.2327 | 7 | 9.9361 | 40 | |
| | 30 | 9.7055 | 22 | 9.7701 | 28 | 0.2299 | 8 | 9.9353 | 30 | |
| | 40 | 9.7076 | 21 | 9.7730 | 29 | 0.2270 | 7 | 9.9346 | 20 | |
| | 50 | 9.7097 | 21 | 9.7759 | 29 | 0.2241 | 8 | 9.9338 | 10 | |
| 31 | 0 | 9.7118 | 21 | 9.7788 | 29 | 0.2212 | 7 | 9.9331 | 0 | 59 |
| | 10 | 9.7139 | 21 | 9.7816 | 28 | 0.2184 | 8 | 9.9323 | 50 | |
| | 20 | 9.7160 | 21 | 9.7845 | 29 | 0.2155 | 8 | 9.9315 | 40 | |
| | 30 | 9.7181 | 21 | 9.7873 | 28 | 0.2127 | 7 | 9.9308 | 30 | |
| | 40 | 9.7201 | 20 | 9.7902 | 29 | 0.2098 | 8 | 9.9300 | 20 | |
| | 50 | 9.7222 | 21 | 9.7930 | 28 | 0.2070 | 8 | 9.9292 | 10 | |
| 32 | 0 | 9.7242 | 20 | 9.7958 | 28 | 0.2042 | 8 | 9.9284 | 0 | 58 |
| | 10 | 9.7262 | 20 | 9.7986 | 28 | 0.2014 | 8 | 9.9276 | 50 | |
| | 20 | 9.7282 | 20 | 9.8014 | 28 | 0.1986 | 8 | 9.9268 | 40 | |
| | 30 | 9.7302 | 20 | 9.8042 | 28 | 0.1958 | 8 | 9.9260 | 30 | |
| | 40 | 9.7322 | 20 | 9.8070 | 27 | 0.1930 | 8 | 9.9252 | 20 | |
| | 50 | 9.7342 | 19 | 9.8097 | 28 | 0.1903 | 8 | 9.9244 | 10 | |
| 33 | 0 | 9.7361 | 19 | 9.8125 | 28 | 0.1875 | 8 | 9.9236 | 0 | 57 |
| | 10 | 9.7380 | 20 | 9.8153 | 27 | 0.1847 | 9 | 9.9228 | 50 | |
| | 20 | 9.7400 | 19 | 9.8180 | 28 | 0.1820 | 8 | 9.9219 | 40 | |
| | 30 | 9.7419 | 19 | 9.8208 | 27 | 0.1792 | 8 | 9.9211 | 30 | |
| | 40 | 9.7438 | 19 | 9.8235 | 28 | 0.1765 | 9 | 9.9203 | 20 | |
| | 50 | 9.7457 | 19 | 9.8263 | 27 | 0.1737 | 8 | 9.9194 | 10 | |
| 34 | 0 | 9.7476 | 18 | 9.8290 | 27 | 0.1710 | 9 | 9.9186 | 0 | 56 |
| | 10 | 9.7494 | 19 | 9.8317 | 27 | 0.1683 | 8 | 9.9177 | 50 | |
| | 20 | 9.7513 | 18 | 9.8344 | 27 | 0.1656 | 9 | 9.9169 | 40 | |
| | 30 | 9.7531 | 19 | 9.8371 | 27 | 0.1629 | 9 | 9.9160 | 30 | |
| | 40 | 9.7550 | 18 | 9.8398 | 27 | 0.1602 | 9 | 9.9151 | 20 | |
| | 50 | 9.7568 | 18 | 9.8425 | 27 | 0.1575 | 8 | 9.9142 | 10 | |
| | | log cos | d | log cot | cd | log tan | d | log sin | ′ | ° |

*Continued*

| ° | ′ | log sin | d | log tan | cd | log cot | d | log cos | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 35 | 0 | 9.7586 | | 9.8452 | | 0.1548 | | 9.9134 | 0 | 55 |
| | 10 | 9.7604 | 18 | 9.8479 | 27 | 0.1521 | 9 | 9.9125 | 50 | |
| | 20 | 9.7622 | 18 | 9.8506 | 27 | 0.1494 | 9 | 9.9116 | 40 | |
| | 30 | 9.7640 | 18 | 9.8533 | 27 | 0.1467 | 9 | 9.9107 | 30 | |
| | 40 | 9.7657 | 17 | 9.8559 | 26 | 0.1441 | 9 | 9.9098 | 20 | |
| | 50 | 9.7675 | 18 | 9.8586 | 27 | 0.1414 | 9 | 9.9089 | 10 | |
| | | | 17 | | 27 | | | | | |
| 36 | 0 | 9.7692 | | 9.8613 | | 0.1387 | | 9.9080 | 0 | 54 |
| | 10 | 9.7710 | 18 | 9.8639 | 26 | 0.1361 | 10 | 9.9070 | 50 | |
| | 20 | 9.7727 | 17 | 9.8666 | 27 | 0.1334 | 9 | 9.9061 | 40 | |
| | 30 | 9.7744 | 17 | 9.8692 | 26 | 0.1308 | 9 | 9.9052 | 30 | |
| | 40 | 9.7761 | 17 | 9.8718 | 26 | 0.1282 | 10 | 9.9042 | 20 | |
| | 50 | 9.7778 | 17 | 9.8745 | 27 | 0.1255 | 9 | 9.9033 | 10 | |
| | | | | | 26 | | 10 | | | |
| 37 | 0 | 9.7795 | | 9.8771 | | 0.1229 | | 9.9023 | 0 | 53 |
| | 10 | 9.7811 | 16 | 9.8797 | 26 | 0.1203 | 9 | 9.9014 | 50 | |
| | 20 | 9.7828 | 17 | 9.8824 | 27 | 0.1176 | 10 | 9.9004 | 40 | |
| | 30 | 9.7844 | 16 | 9.8850 | 26 | 0.1150 | 9 | 9.8995 | 30 | |
| | 40 | 9.7861 | 17 | 9.8876 | 26 | 0.1124 | 10 | 9.8985 | 20 | |
| | 50 | 9 7877 | 16 | 9.8902 | 26 | 0.1098 | 10 | 9.8975 | 10 | |
| | | | 16 | | 26 | | 10 | | | |
| 38 | 0 | 9.7893 | | 9.8928 | | 0.1072 | | 9.8965 | 0 | 52 |
| | 10 | 9.7910 | 17 | 9.8954 | 26 | 0.1046 | 11 | 9.8955 | 50 | |
| | 20 | 9.7926 | 16 | 9.8980 | 26 | 0.1020 | 10 | 9.8945 | 40 | |
| | 30 | 9.7941 | 15 | 9.9006 | 26 | 0.0994 | 10 | 9.8935 | 30 | |
| | 40 | 9.7957 | 16 | 9.9032 | 26 | 0.0968 | 10 | 9.8925 | 20 | |
| | 50 | 9.7973 | 16 | 9.9058 | 26 | 0.0942 | 10 | 9.8915 | 10 | |
| | | | 16 | | 26 | | 10 | | | |
| 39 | 0 | 9.7989 | | 9.9084 | | 0.0916 | | 9.8905 | 0 | 51 |
| | 10 | 9.8004 | 15 | 9.9110 | 26 | 0.0890 | 10 | 9.8895 | 50 | |
| | 20 | 9.8020 | 16 | 9.9135 | 25 | 0.0865 | 11 | 9.8884 | 40 | |
| | 30 | 9.8035 | 15 | 9.9161 | 26 | 0.0839 | 10 | 9.8874 | 30 | |
| | 40 | 9.8050 | 15 | 9.9187 | 26 | 0.0813 | 10 | 9.8864 | 20 | |
| | 50 | 9.8066 | 16 | 9.9212 | 25 | 0.0788 | 11 | 9.8853 | 10 | |
| | | | 15 | | 26 | | 10 | | | |
| | | log cos | d | log cot | cd | log tan | d | log sin | ′ | ° |

| ° | ′ | log sin | d | log tan | cd | log cot | d | log cos | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 40 | 0 | 9.8081 | | 9.9238 | | 0.0762 | | 9.8843 | 0 | 50 |
| | 10 | 9.8096 | 15 | 9.9264 | 26 | 0.0736 | 11 | 9.8832 | 50 | |
| | 20 | 9.8111 | 15 | 9.9289 | 25 | 0.0711 | 11 | 9.8821 | 40 | |
| | 30 | 9.8125 | 14 | 9.9315 | 26 | 0.0685 | 11 | 9.8810 | 30 | |
| | 40 | 9.8140 | 15 | 9.9341 | 26 | 0.0659 | 10 | 9.8800 | 20 | |
| | 50 | 9.8155 | 15 | 9.9366 | 25 | 0.0634 | 11 | 9.8789 | 10 | |
| | | | 14 | | 26 | | 11 | | | |
| 41 | 0 | 9.8169 | | 9.9392 | | 0.0608 | | 9.8778 | 0 | 49 |
| | 10 | 9.8184 | 15 | 9.9417 | 25 | 0.0583 | 11 | 9.8767 | 50 | |
| | 20 | 9.8198 | 14 | 9.9443 | 26 | 0.0557 | 11 | 9.8756 | 40 | |
| | 30 | 9.8213 | 15 | 9.9468 | 25 | 0.0532 | 11 | 9.8745 | 30 | |
| | 40 | 9.8227 | 14 | 9.9494 | 26 | 0.0506 | 12 | 9.8733 | 20 | |
| | 50 | 9.8241 | 14 | 9.9519 | 25 | 0.0481 | 11 | 9.8722 | 10 | |
| | | | 14 | | 25 | | 11 | | | |
| 42 | 0 | 9.8255 | | 9.9544 | | 0.0456 | | 9.8711 | 0 | 48 |
| | 10 | 9.8269 | 14 | 9.9570 | 26 | 0.0430 | 12 | 9.8699 | 50 | |
| | 20 | 9.8283 | 14 | 9.9595 | 25 | 0.0405 | 11 | 9.8688 | 40 | |
| | 30 | 9.8297 | 14 | 9.9621 | 26 | 0.0379 | 12 | 9.8676 | 30 | |
| | 40 | 9.8311 | 14 | 9.9646 | 25 | 0.0354 | 11 | 9.8665 | 20 | |
| | 50 | 9.8324 | 13 | 9.9671 | 25 | 0.0329 | 12 | 9.8653 | 10 | |
| | | | 14 | | 26 | | 12 | | | |
| 43 | 0 | 9.8338 | | 9.9697 | | 0.0303 | | 9.8641 | 0 | 47 |
| | 10 | 9.8351 | 13 | 9.9722 | 25 | 0.0278 | 12 | 9.8629 | 50 | |
| | 20 | 9.8365 | 14 | 9.9747 | 25 | 0.0253 | 11 | 9.8618 | 40 | |
| | 30 | 9.8378 | 13 | 9.9772 | 25 | 0.0228 | 12 | 9.8606 | 30 | |
| | 40 | 9.8391 | 13 | 9.9798 | 26 | 0.0202 | 12 | 9.8594 | 20 | |
| | 50 | 9.8405 | 14 | 9.9823 | 25 | 0.0177 | 12 | 9.8582 | 10 | |
| | | | 13 | | 25 | | 13 | | | |
| 44 | 0 | 9.8418 | | 9.9848 | | 0.0152 | | 9.8569 | 0 | 46 |
| | 10 | 9.8431 | 13 | 9.9874 | 26 | 0.0126 | 12 | 9.8557 | 50 | |
| | 20 | 9.8444 | 13 | 9.9899 | 25 | 0.0101 | 12 | 9.8545 | 40 | |
| | 30 | 9.8457 | 13 | 9.9924 | 25 | 0.0076 | 13 | 9.8532 | 30 | |
| | 40 | 9.8469 | 12 | 9.9949 | 25 | 0.0051 | 12 | 9.8520 | 20 | |
| | 50 | 9.8482 | 13 | 9.9975 | 26 | 0.0025 | 13 | 9.8507 | 10 | |
| | | | 13 | | 25 | | 12 | | | |
| 45 | 0 | 9.8495 | | 10.0000 | | 0.0000 | | 9.8495 | | 45 |
| | | log cos | d | log cot | cd | log tan | d | log sin | ′ | ° |

## 6. Sines and Cosines (see Secs. 183, 184 on how to use the tables)

SINES

| Deg. | 0′ → | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | | Proportional parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ |
| 0 | 0.0000 | 0.0029 | 0.0058 | 0.0087 | 0.0116 | 0.0145 | 0.0175 | 89 | 3 | 6 | 9 | 12 | 15 | 17 | 20 | 23 | 26 |
| 1 | 0.0175 | 0.0204 | 0.0233 | 0.0262 | 0.0291 | 0.0320 | 0.0349 | 88 | 3 | 6 | 9 | 12 | 15 | 17 | 20 | 23 | 26 |
| 2 | 0.0349 | 0.0378 | 0.0407 | 0.0436 | 0.0465 | 0.0494 | 0.0523 | 87 | 3 | 6 | 9 | 12 | 15 | 17 | 20 | 23 | 26 |
| 3 | 0.0523 | 0.0552 | 0.0581 | 0.0610 | 0.0640 | 0.0669 | 0.0698 | 86 | 3 | 6 | 9 | 12 | 15 | 17 | 20 | 23 | 26 |
| 4 | 0.0698 | 0.0727 | 0.0756 | 0.0785 | 0.0814 | 0.0843 | 0.0872 | 85 | 3 | 6 | | 12 | 15 | 17 | 20 | 23 | 26 |
| | | | | | | | | | | | 9 | | | | | | |
| 5 | 0.0872 | 0.0901 | 0.0929 | 0.0958 | 0.0987 | 0.1016 | 0.1045 | 84 | 3 | 6 | 9 | 12 | 14 | 17 | 20 | 23 | 26 |
| 6 | 0.1045 | 0.1074 | 0.1103 | 0.1132 | 0.1161 | 0.1190 | 0.1219 | 83 | 3 | 6 | 9 | 12 | 14 | 17 | 20 | 23 | 26 |
| 7 | 0.1219 | 0.1248 | 0.1276 | 0.1305 | 0.1334 | 0.1363 | 0.1392 | 82 | 3 | 6 | 9 | 12 | 14 | 17 | 20 | 23 | 26 |
| 8 | 0.1392 | 0.1421 | 0.1449 | 0.1478 | 0.1507 | 0.1536 | 0.1564 | 81 | 3 | 6 | 9 | 12 | 14 | 17 | 20 | 23 | 26 |
| 9 | 0.1564 | 0.1593 | 0.1622 | 0.1650 | 0.1679 | 0.1708 | 0.1736 | 80 | 3 | 6 | 9 | 12 | 14 | 17 | 20 | 23 | 26 |
| 10 | 0.1736 | 0.1765 | 0.1794 | 0.1822 | 0.1851 | 0.1880 | 0.1908 | 79 | 3 | 6 | 9 | 11 | 14 | 17 | 20 | 23 | 26 |
| 11 | 0.1908 | 0.1937 | 0.1965 | 0.1994 | 0.2022 | 0.2051 | 0.2079 | 78 | 3 | 6 | 9 | 11 | 14 | 17 | 20 | 23 | 26 |
| 12 | 0.2079 | 0.2108 | 0.2136 | 0.2164 | 0.2193 | 0.2221 | 0.2250 | 77 | 3 | 6 | 9 | 11 | 14 | 17 | 20 | 23 | 26 |
| 13 | 0.2250 | 0.2278 | 0.2306 | 0.2334 | 0.2363 | 0.2391 | 0.2419 | 76 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 23 | 25 |
| 14 | 0.2419 | 0.2447 | 0.2476 | 0.2504 | 0.2532 | 0.2560 | 0.2588 | 75 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 23 | 25 |
| 15 | 0.2588 | 0.2616 | 0.2644 | 0.2672 | 0.2700 | 0.2728 | 0.2756 | 74 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 22 | 25 |
| 16 | 0.2756 | 0.2784 | 0.2812 | 0.2840 | 0.2868 | 0.2896 | 0.2924 | 73 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 22 | 25 |
| 17 | 0.2924 | 0.2952 | 0.2979 | 0.3007 | 0.3035 | 0.3062 | 0.3090 | 72 | 3 | 6 | 8 | 11 | 14 | 17 | 19 | 22 | 25 |
| 18 | 0.3090 | 0.3118 | 0.3145 | 0.3173 | 0.3201 | 0.3228 | 0.3256 | 71 | 3 | 6 | 8 | 11 | 14 | 17 | 19 | 22 | 25 |
| 19 | 0.3256 | 0.3283 | 0.3311 | 0.3338 | 0.3365 | 0.3393 | 0.3420 | 70 | 3 | 5 | 8 | 11 | 14 | 16 | 19 | 22 | 25 |
| 20 | 0.3420 | 0.3448 | 0.3475 | 0.3502 | 0.3529 | 0.3557 | 0.3584 | 69 | 3 | 5 | 8 | 11 | 14 | 16 | 19 | 22 | 25 |
| 21 | 0.3584 | 0.3611 | 0.3638 | 0.3665 | 0.3692 | 0.3719 | 0.3746 | 68 | 3 | 5 | 8 | 11 | 14 | 16 | 19 | 22 | 24 |

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 22 | 0.3746 | 0.3773 | 0.3800 | 0.3827 | 0.3854 | 0.3881 | 0.3907 | 67 | 3 | 5 | 8 | 11 | 13 | 16 | 19 | 22 | 24 |
| 23 | 0.3907 | 0.3934 | 0.3961 | 0.3987 | 0.4014 | 0.4041 | 0.4067 | 66 | 3 | 5 | 8 | 11 | 13 | 16 | 19 | 21 | 24 |
| 24 | 0.4067 | 0.4094 | 0.4120 | 0.4147 | 0.4173 | 0.4200 | 0.4226 | 65 | 3 | 5 | 8 | 11 | 13 | 16 | 19 | 21 | 24 |
| 25 | 0.4226 | 0.4253 | 0.4279 | 0.4305 | 0.4331 | 0.4358 | 0.4384 | 64 | 3 | 5 | 8 | 10 | 13 | 16 | 18 | 21 | 24 |
| 26 | 0.4384 | 0.4410 | 0.4436 | 0.4462 | 0.4488 | 0.4514 | 0.4540 | 63 | 3 | 5 | 8 | 10 | 13 | 16 | 18 | 21 | 23 |
| 27 | 0.4540 | 0.4566 | 0.4592 | 0.4617 | 0.4643 | 0.4669 | 0.4695 | 62 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 21 | 23 |
| 28 | 0.4695 | 0.4720 | 0.4746 | 0.4772 | 0.4797 | 0.4823 | 0.4848 | 61 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 |
| 29 | 0.4848 | 0.4874 | 0.4899 | 0.4924 | 0.4950 | 0.4975 | 0.5000 | 60 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 |
| 30 | 0.5000 | 0.5025 | 0.5050 | 0.5075 | 0.5100 | 0.5125 | 0.5150 | 59 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 |
| 31 | 0.5150 | 0.5175 | 0.5200 | 0.5225 | 0.5250 | 0.5275 | 0.5299 | 58 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 |
| 32 | 0.5299 | 0.5324 | 0.5348 | 0.5373 | 0.5398 | 0.5422 | 0.5446 | 57 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 |
| 33 | 0.5446 | 0.5471 | 0.5495 | 0.5519 | 0.5544 | 0.5568 | 0.5592 | 56 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 19 | 22 |
| 34 | 0.5592 | 0.5616 | 0.5640 | 0.5664 | 0.5688 | 0.5712 | 0.5736 | 55 | 2 | 5 | 7 | 10 | 12 | 14 | 17 | 19 | 22 |
| 35 | 0.5736 | 0.5760 | 0.5783 | 0.5807 | 0.5831 | 0.5854 | 0.5878 | 54 | 2 | 5 | 7 | 9 | 12 | 14 | 17 | 19 | 21 |
| 36 | 0.5878 | 0.5901 | 0.5925 | 0.5948 | 0.5972 | 0.5995 | 0.6018 | 53 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 19 | 21 |
| 37 | 0.6018 | 0.6041 | 0.6065 | 0.6088 | 0.6111 | 0.6134 | 0.6157 | 52 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 18 | 21 |
| 38 | 0.6157 | 0.6180 | 0.6202 | 0.6225 | 0.6248 | 0.6271 | 0.6293 | 51 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |
| 39 | 0.6293 | 0.6316 | 0.6338 | 0.6361 | 0.6383 | 0.6406 | 0.6428 | 50 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| 40 | 0.6428 | 0.6450 | 0.6472 | 0.6494 | 0.6517 | 0.6539 | 0.6561 | 49 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 18 | 20 |
| 41 | 0.6561 | 0.6583 | 0.6604 | 0.6626 | 0.6648 | 0.6670 | 0.6691 | 48 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| 42 | 0.6691 | 0.6713 | 0.6734 | 0.6756 | 0.6777 | 0.6799 | 0.6820 | 47 | 2 | 4 | 6 | 9 | 11 | 13 | 15 | 17 | 19 |
| 43 | 0.6820 | 0.6841 | 0.6862 | 0.6884 | 0.6905 | 0.6926 | 0.6947 | 46 | 2 | 4 | 6 | 9 | 11 | 13 | 15 | 17 | 19 |
| 44 | 0.6947 | 0.6967 | 0.6988 | 0.7009 | 0.7030 | 0.7050 | 0.7071 | 45 | 2 | 4 | 6 | 9 | 10 | 12 | 15 | 17 | 19 |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ ← | Deg. | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ |

| Deg. | 0′ → | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | | Proportional parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ |
| 45 | 0.7071 | 0.7092 | 0.7112 | 0.7133 | 0.7153 | 0.7173 | 0.7193 | 44 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 46 | 0.7193 | 0.7214 | 0.7234 | 0.7254 | 0.7274 | 0.7294 | 0.7314 | 43 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 47 | 0.7314 | 0.7333 | 0.7353 | 0.7373 | 0.7392 | 0.7412 | 0.7431 | 42 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 48 | 0.7431 | 0.7451 | 0.7470 | 0.7490 | 0.7509 | 0.7528 | 0.7547 | 41 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 49 | 0.7547 | 0.7566 | 0.7585 | 0.7604 | 0.7623 | 0.7642 | 0.7660 | 40 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| 50 | 0.7660 | 0.7679 | 0.7698 | 0.7716 | 0.7735 | 0.7753 | 0.7771 | 39 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 51 | 0.7771 | 0.7790 | 0.7808 | 0.7826 | 0.7844 | 0.7862 | 0.7880 | 38 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| 52 | 0.7880 | 0.7898 | 0.7916 | 0.7934 | 0.7951 | 0.7969 | 0.7986 | 37 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 53 | 0.7986 | 0.8004 | 0.8021 | 0.8039 | 0.8056 | 0.8073 | 0.8090 | 36 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| 54 | 0.8090 | 0.8107 | 0.8124 | 0.8141 | 0.8158 | 0.8175 | 0.8192 | 35 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| 55 | 0.8192 | 0.8208 | 0.8225 | 0.8241 | 0.8258 | 0.8274 | 0.8290 | 34 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| 56 | 0.8290 | 0.8307 | 0.8323 | 0.8339 | 0.8355 | 0.8371 | 0.8387 | 33 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 |
| 57 | 0.8387 | 0.8403 | 0.8418 | 0.8434 | 0.8450 | 0.8465 | 0.8480 | 32 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 58 | 0.8480 | 0.8496 | 0.8511 | 0.8526 | 0.8542 | 0.8557 | 0.8572 | 31 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 59 | 0.8572 | 0.8587 | 0.8601 | 0.8616 | 0.8631 | 0.8646 | 0.8660 | 30 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 60 | 0.8660 | 0.8675 | 0.8689 | 0.8704 | 0.8718 | 0.8732 | 0.8746 | 29 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 61 | 0.8746 | 0.8760 | 0.8774 | 0.8788 | 0.8802 | 0.8816 | 0.8829 | 28 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| 62 | 0.8829 | 0.8843 | 0.8857 | 0.8870 | 0.0884 | 0.8897 | 0.8910 | 27 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 63 | 0.8910 | 0.8923 | 0.8936 | 0.8949 | 0.8962 | 0.8975 | 0.8988 | 26 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 64 | 0.8988 | 0.9001 | 0.9013 | 0.9026 | 0.9038 | 0.9051 | 0.9063 | 25 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 65 | 0.9063 | 0.9075 | 0.9088 | 0.9100 | 0.9112 | 0.9124 | 0.9135 | 24 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 66 | 0.9135 | 0.9147 | 0.9159 | 0.9171 | 0.9182 | 0.9194 | 0.9205 | 23 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 67 | 0.9205 | 0.9216 | 0.9228 | 0.9239 | 0.9250 | 0.9261 | 0.9272 | 22 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| 68 | 0.9272 | 0.9283 | 0.9293 | 0.9304 | 0.9315 | 0.9325 | 0 9336 | 21 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| 69 | 0.9336 | 0.9346 | 0.9356 | 0.9367 | 0.9377 | 0.9387 | 0.9397 | 20 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 70 | 0.9397 | 0.9407 | 0.9417 | 0 9426 | 0.9436 | 0.9446 | 0.9455 | 19 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 71 | 0.9455 | 0.9465 | 0.9474 | 0.9483 | 0.9492 | 0.9502 | 0.9511 | 18 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| 72 | 0.9511 | 0.9520 | 0.9528 | 0.9537 | 0.9546 | 0.9555 | 0.9563 | 17 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 73 | 0.9563 | 0.9572 | 0.9580 | 0.9588 | 0.9596 | 0.9605 | 0.9613 | 16 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| 74 | 0.9613 | 0.9621 | 0.9628 | 0.9636 | 0.9644 | 0.9652 | 0.9659 | 15 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 75 | 0.9659 | 0.9667 | 0.9674 | 0.9681 | 0.9689 | 0.9696 | 0.9703 | 14 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 76 | 0.9703 | 0.9710 | 0.9717 | 0.9724 | 0.9730 | 0.9737 | 0.9744 | 13 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 77 | 0.9744 | 0.9750 | 0.9757 | 0.9763 | 0.9769 | 0.9775 | 0.9781 | 12 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 78 | 0.9781 | 0.9787 | 0.9793 | 0.9799 | 0.9805 | 0.9811 | 0.9816 | 11 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 79 | 0.9816 | 0.9822 | 0.9827 | 0.9833 | 0.9838 | 0.9843 | 0.9848 | 10 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 80 | 0.9848 | 0.9853 | 0.9858 | 0.9863 | 0.9868 | 0.9872 | 0.9877 | 9 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 81 | 0.9877 | 0.9881 | 0.9886 | 0.9890 | 0.9894 | 0.9899 | 0.9903 | 8 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 82 | 0.9903 | 0.9907 | 0.9911 | 0.9914 | 0 9918 | 0.9922 | 0.9925 | 7 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| 83 | 0.9925 | 0.9929 | 0.9932 | 0.9936 | 0.9939 | 0.9942 | 0.9945 | 6 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 84 | 0.9945 | 0.9948 | 0.9951 | 0.9954 | 0.9957 | 0.9959 | 0 9962 | 5 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 85 | 0.9962 | 0.9964 | 0.9967 | 0.9969 | 0.9971 | 0.9974 | 0.9976 | 4 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 86 | 0.9976 | 0.9978 | 0.9980 | 0.9981 | 0.9983 | 0.9985 | 0.9986 | 3 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |
| 87 | 0.9986 | 0.9988 | 0.9989 | 0.9990 | 0.9992 | 0.9993 | 0.9994 | 2 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 88 | 0.9994 | 0.9995 | 0.9996 | 0.9997 | 0.9997 | 0.9998 | 0 9998 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 89 | 0.9998 | 0.9999 | 0.9999 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ ← | Deg. | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ |

## 7. Tangents and Cotangents (see Secs. 183, 184)

TANGENTS

| Deg. | 0′ → | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | | Proportional parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ |
| 0↓ | 0.0000 | 0.0029 | 0.0058 | 0.0087 | 0.0116 | 0.0145 | 0.0175 | 89 | 3 | 6 | 9 | 12 | 15 | 17 | 20 | 23 | 26 |
| 1 | 0.0175 | 0.0204 | 0.0233 | 0.0262 | 0.0291 | 0.0320 | 0.0349 | 88 | 3 | 6 | 9 | 12 | 15 | 17 | 20 | 23 | 26 |
| 2 | 0.0349 | 0.0378 | 0.0407 | 0.0437 | 0.0466 | 0.0495 | 0.0524 | 87 | 3 | 6 | 9 | 12 | 15 | 17 | 20 | 23 | 26 |
| 3 | 0.0524 | 0.0553 | 0.0582 | 0.0612 | 0.0641 | 0.0670 | 0.0699 | 86 | 3 | 6 | 9 | 12 | 15 | 18 | 20 | 23 | 26 |
| 4 | 0.0699 | 0.0729 | 0.0758 | 0 0787 | 0.0816 | 0.0846 | 0.0875 | 85 | 3 | 6 | 9 | 12 | 15 | 18 | 20 | 23 | 26 |
| 5 | 0.0875 | 0.0904 | 0.0934 | 0.0963 | 0.0992 | 0.1022 | 0.1051 | 84 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 23 | 26 |
| 6 | 0.1051 | 0.1080 | 0.1110 | 0.1139 | 0.1169 | 0.1198 | 0.1228 | 83 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 7 | 0.1228 | 0.1257 | 0.1287 | 0.1317 | 0.1346 | 0.1376 | 0.1405 | 82 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 8 | 0.1405 | 0.1435 | 0.1465 | 0.1495 | 0.1524 | 0.1554 | 0.1584 | 81 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 9 | 0.1584 | 0.1614 | 0.1644 | 0.1673 | 0.1703 | 0.1733 | 0.1763 | 80 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 10 | 0.1763 | 0.1793 | 0.1823 | 0.1853 | 0.1883 | 0.1914 | 0.194 | 79 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 11 | 0.1944 | 0.1974 | 0.2004 | 0.2035 | 0.2065 | 0.2095 | 0.2126 | 78 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 12 | 0.2126 | 0.2156 | 0.2186 | 0.2217 | 0.2247 | 0.2278 | 0.2309 | 77 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 13 | 0.2309 | 0.2339 | 0.2370 | 0.2401 | 0.2432 | 0.2462 | 0.2493 | 76 | 3 | 6 | 9 | 12 | 15 | 18 | 22 | 25 | 28 |
| 14 | 0.2493 | 0.2524 | 0.2555 | 0.2586 | 0.2617 | 0.2648 | 0.2679 | 75 | 3 | 6 | 9 | 12 | 16 | 19 | 22 | 25 | 28 |
| 15 | 0.2679 | 0.2711 | 0.2742 | 0.2773 | 0.2805 | 0.2836 | 0.2867 | 74 | 3 | 6 | 9 | 13 | 16 | 19 | 22 | 25 | 28 |
| 16 | 0.2867 | 0.2899 | 0.2931 | 0.2962 | 0.2994 | 0.3026 | 0.3057 | 73 | 3 | 6 | 9 | 13 | 16 | 19 | 22 | 25 | 28 |
| 17 | 0.3057 | 0.3089 | 0.3121 | 0.3153 | 0.3185 | 0.3217 | 0.3249 | 72 | 3 | 6 | 10 | 13 | 16 | 19 | 22 | 26 | 29 |
| 18 | 0.3249 | 0.3281 | 0.3314 | 0.3346 | 0.3378 | 0.3411 | 0.3443 | 71 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 19 | 0.3443 | 0.3476 | 0.3508 | 0.3541 | 0.3574 | 0.3607 | 0.3640 | 70 | 3 | 7 | 10 | 13 | 16 | 20 | 23 | 26 | 29 |
| 20 | 0.3640 | 0.3673 | 0.3706 | 0.3739 | 0.3772 | 0.3805 | 0.3839 | 69 | 3 | 7 | 10 | 13 | 17 | 20 | 23 | 27 | 30 |
| 21 | 0.3839 | 0.3872 | 0.3906 | 0.3939 | 0.3973 | 0.4006 | 0.4040 | 68 | 3 | 7 | 10 | 13 | 17 | 20 | 24 | 27 | 30 |

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 22 | 0.4040 | 0.4074 | 0.4108 | 0.4142 | 0.4176 | 0.4210 | 0.4245 | 67 | 3 | 7 | 10 | 14 | 17 | 20 | 24 | 27 | 31 |
| 23 | 0.4245 | 0.4279 | 0.4314 | 0.4348 | 0.4383 | 0.4417 | 0.4452 | 66 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 28 | 31 |
| 24 | 0.4452 | 0.4487 | 0.4522 | 0.4557 | 0.4592 | 0.4628 | 0.4663 | 65 | 4 | 7 | 11 | 14 | 18 | 21 | 25 | 28 | 32 |
| 25 | 0.4663 | 0.4699 | 0.4734 | 0.4770 | 0.4806 | 0.4841 | 0.4877 | 64 | 4 | 7 | 11 | 14 | 18 | 21 | 25 | 29 | 32 |
| 26 | 0.4877 | 0.4913 | 0.4950 | 0.4986 | 0.5022 | 0.5059 | 0.5095 | 63 | 4 | 7 | 11 | 15 | 18 | 22 | 25 | 29 | 33 |
| 27 | 0.5095 | 0.5132 | 0.5169 | 0.5206 | 0.5243 | 0.5280 | 0.5317 | 62 | 4 | 7 | 11 | 15 | 18 | 22 | 26 | 29 | 33 |
| 28 | 0.5317 | 0.5354 | 0.5392 | 0.5430 | 0.5467 | 0.5505 | 0.5543 | 61 | 4 | 8 | 11 | 15 | 19 | 23 | 26 | 30 | 34 |
| 29 | 0.5543 | 0.5581 | 0.5619 | 0.5658 | 0.5696 | 0.5735 | 0.5774 | 60 | 4 | 8 | 12 | 15 | 19 | 23 | 27 | 31 | 35 |
| 30 | 0.5774 | 0.5812 | 0.5851 | 0.5890 | 0.5930 | 0.5969 | 0.6009 | 59 | 4 | 8 | 12 | 16 | 20 | 24 | 27 | 31 | 35 |
| 31 | 0.6009 | 0.6048 | 0.6088 | 0.6128 | 0.6168 | 0.6208 | 0.6249 | 58 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 32 | 0.6249 | 0.6289 | 0.6330 | 0.6371 | 0.6412 | 0.6453 | 0.6494 | 57 | 4 | 8 | 12 | 16 | 20 | 25 | 29 | 33 | 37 |
| 33 | 0.6494 | 0.6536 | 0.6577 | 0.6619 | 0.6661 | 0.6703 | 0.6745 | 56 | 4 | 8 | 13 | 17 | 21 | 25 | 29 | 33 | 38 |
| 34 | 0.6745 | 0.6787 | 0.6830 | 0.6873 | 0.6916 | 0.6959 | 0.7002 | 55 | 4 | 9 | 13 | 17 | 21 | 26 | 30 | 34 | 39 |
| 35 | 0.7002 | 0.7046 | 0.7089 | 0.7133 | 0.7177 | 0.7221 | 0.7265 | 54 | 4 | 9 | 13 | 18 | 22 | 26 | 31 | 35 | 39 |
| 36 | 0.7265 | 0.7310 | 0.7355 | 0.7400 | 0.7445 | 0.7490 | 0.7536 | 53 | 5 | 9 | 14 | 18 | 23 | 27 | 32 | 36 | 41 |
| 37 | 0.7536 | 0.7581 | 0.7627 | 0.7673 | 0.7720 | 0.7766 | 0.7813 | 52 | 5 | 9 | 14 | 19 | 23 | 28 | 32 | 37 | 42 |
| 38 | 0.7813 | 0.7860 | 0.7907 | 0.7954 | 0.8002 | 0.8050 | 0.8098 | 51 | 5 | 9 | 14 | 19 | 24 | 28 | 33 | 38 | 43 |
| 39 | 0.8098 | 0.8146 | 0.8195 | 0.8243 | 0.8292 | 0.8342 | 0.8391 | 50 | 5 | 10 | 15 | 20 | 24 | 29 | 34 | 39 | 44 |
| 40 | 0.8391 | 0.8441 | 0.8491 | 0.8541 | 0.8591 | 0.8642 | 0.8693 | 49 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 |
| 41 | 0.8693 | 0.8744 | 0.8796 | 0.8847 | 0.8899 | 0.8952 | 0.9004 | 48 | 5 | 10 | 16 | 21 | 26 | 31 | 36 | 41 | 47 |
| 42 | 0.9004 | 0.9057 | 0.9110 | 0.9163 | 0.9217 | 0.9271 | 0.9325 | 47 | 5 | 11 | 16 | 21 | 27 | 32 | 38 | 43 | 48 |
| 43 | 0.9325 | 0.9380 | 0.9435 | 0.9490 | 0.9545 | 0.9601 | 0.9657 | 46 | 6 | 11 | 17 | 22 | 28 | 33 | 39 | 44 | 50 |
| 44 | 0.9657 | 0.9713 | 0.9770 | 0.9827 | 0.9884 | 0.9942 | 1.0000 | †45 | 6 | 11 | 17 | 23 | 29 | 34 | 40 | 46 | 51 |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ ← | Deg. | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ |

## TANGENTS *Continued*

| Deg. | 0′ → | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | | Proportional parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ |
| 45 | 1.0000 | 1.0058 | 1.0117 | 1.0176 | 1.0236 | 1.0295 | 1.0355 | 44 | 6 | 12 | 18 | 24 | 30 | 36 | 41 | 47 | 53 |
| 46 | 1.0355 | 1.0416 | 1.0477 | 1.0538 | 1.0599 | 1.0661 | 1.0724 | 43 | 6 | 12 | 18 | 25 | 31 | 37 | 43 | 49 | 55 |
| 47 | 1.0724 | 1.0786 | 1.0850 | 1.0913 | 1.0977 | 1.1041 | 1.1106 | 42 | 6 | 13 | 19 | 25 | 32 | 38 | 45 | 51 | 57 |
| 48 | 1.1106 | 1.1171 | 1.1237 | 1.1303 | 1.1369 | 1.1436 | 1.1504 | 41 | 7 | 13 | 20 | 27 | 33 | 40 | 46 | 53 | 59 |
| 49 | 1.1504 | 1.1572 | 1.1640 | 1.1708 | 1.1778 | 1.1847 | 1.1918 | 40 | 7 | 14 | 21 | 28 | 34 | 41 | 48 | 55 | 62 |
| 50 | 1.1918 | 1.1988 | 1.2059 | 1.2131 | 1.2203 | 1.2276 | 1.2349 | 39 | 7 | 14 | 22 | 29 | 36 | 43 | 50 | 57 | 65 |
| 51 | 1.2349 | 1.2423 | 1.2497 | 1.2572 | 1.2647 | 1.2723 | 1.2799 | 38 | 8 | 15 | 23 | 30 | 38 | 45 | 53 | 60 | 68 |
| 52 | 1.2799 | 1.2876 | 1.2954 | 1.3032 | | | | | 8 | 16 | 23 | 31 | 39 | 47 | 55 | 62 | 70 |
| | | | | | 1.3111 | 1.3190 | 1.3270 | 37 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 |
| 53 | 1.3270 | 1.3351 | 1.3432 | 1.3514 | | | | | 8 | 16 | 24 | 32 | 41 | 49 | 57 | 65 | 73 |
| | | | | | 1.3597 | 1.3680 | 1.3764 | 36 | 8 | 17 | 25 | 33 | 42 | 50 | 58 | 67 | 75 |
| 54 | 1.3764 | 1.3848 | 1.3937 | 1.4020 | | | | | 9 | 17 | 26 | 34 | 43 | 51 | 60 | 68 | 77 |
| | | | | | 1.4106 | 1.4193 | 1.4282 | 35 | 9 | 17 | 26 | 35 | 44 | 52 | 61 | 70 | 79 |
| 55 | 1.4282 | 1.4370 | 1.4460 | 1.4550 | | | | | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 |
| | | | | | 1.4641 | 1.4733 | 1.4826 | 34 | 9 | 18 | 28 | 37 | 46 | 55 | 64 | 74 | 83 |
| 56 | 1.4826 | 1.4919 | 1.5013 | 1.5108 | | | | | 9 | 19 | 28 | 38 | 47 | 56 | 66 | 75 | 85 |
| | | | | | 1.5204 | 1.5301 | 1.5399 | 33 | 10 | 19 | 29 | 39 | 48 | 58 | 68 | 78 | 87 |
| 57 | 1.5399 | 1.5497 | 1.5597 | 1.5697 | | | | | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
| | | | | | 1.5798 | 1.5900 | 1.6003 | 32 | 10 | 20 | 31 | 41 | 51 | 61 | 71 | 82 | 92 |
| 58 | 1.6003 | 1.6107 | 1.6212 | 1.6318 | | | | | 10 | 21 | 31 | 42 | 52 | 63 | 73 | 84 | 94 |
| | | | | | 1 6426 | 1.6534 | 1.6643 | 31 | 11 | 22 | 32 | 43 | 54 | 65 | 76 | 86 | 97 |
| 59 | 1.6643 | 1.6753 | 1.6864 | 1.6977 | | | | | 11 | 22 | 33 | 44 | 55 | 67 | 78 | 89 | 100 |
| | | | | | 1.7090 | 1.7205 | 1.7320 | 30 | 11 | 23 | 34 | 46 | 57 | 69 | 80 | 92 | 103 |
| 60 | 1.7320 | 1.7438 | 1.7556 | 1.7675 | | | | | 12 | 24 | 35 | 47 | 59 | 71 | 83 | 94 | 106 |
| | | | | | 1.7796 | 1.7917 | 1.8040 | 29 | 12 | 24 | 37 | 49 | 61 | 73 | 85 | 98 | 110 |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 61 | 1.804 | 1.816 | 1.829 | 1.842 | 1.855 | 1.868 | 1.881 | 28 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 62 | 1.881 | 1.894 | 1.907 | 1.921 | 1.935 | 1.949 | 1.963 | 27 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 11 |
| 63 | 1.963 | 1.977 | 1.991 | 2.006 | 2.020 | 2.035 | 2.050 | 26 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 12 |
| 64 | 2.050 | 2.066 | 2.081 | 2.097 | 2.112 | 2.128 | 2.145 | 25 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 65 | 2.145 | 2.161 | 2.177 | 2.194 | 2.211 | 2.229 | 2.246 | 24 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 14 | 15 |
| 66 | 2.246 | 2.264 | 2.282 | 2.300 | 2.318 | 2.337 | 2.356 | 23 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 15 | 16 |
| 67 | 2.356 | 2.375 | 2.394 | 2.414 | 2.434 | 2.455 | 2.475 | 22 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 68 | 2.475 | 2.496 | 2.517 | 2.539 | 2.560 | 2.583 | 2.605 | 21 | 2 | 4 | 6 | 9 | 11 | 13 | 15 | 17 | 19 |
| 69 | 2.605 | 2.628 | 2.651 | 2.675 | 2.699 | 2.723 | 2.747 | 20 | 2 | 5 | 7 | 9 | 12 | 14 | 17 | 19 | 21 |
| 70 | 2.747 | 2.773 | 2.798 | 2.824 | 2.850 | 2.877 | 2.904 | 19 | 3 | 5 | 8 | 10 | 13 | 16 | 18 | 21 | 24 |
| 71 | 2.904 | 2.932 | 2.960 | 2.989 | 3.018 | 3.047 | 3.078 | 18 | 3 | 6 | 9 | 12 | 14 | 17 | 20 | 23 | 26 |
| 72 | 3.078 | 3.108 | 3.140 | 3.172 | | | | | 3 | 6 | 9 | 13 | 16 | 19 | 22 | 25 | 28 |
| | | | | | 3.204 | 3.237 | 3.271 | 17 | 3 | 7 | 10 | 13 | 17 | 20 | 23 | 26 | 30 |
| 73 | 3.271 | 3.305 | 3.340 | 3.376 | | | | | 4 | 7 | 11 | 14 | 18 | 21 | 25 | 28 | 31 |
| | | | | | 3.412 | 3.450 | 3.487 | 16 | 4 | 7 | 11 | 15 | 19 | 22 | 26 | 30 | 33 |
| 74 | 3.487 | 3.526 | 3.566 | 3.606 | | | | | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| | | | | | 3.647 | 3.689 | 3.732 | 15 | 4 | 8 | 13 | 17 | 21 | 25 | 29 | 34 | 38 |
| 75 | 3.732 | 3.776 | 3.821 | 3.867 | | | | | 4 | 9 | 13 | 18 | 22 | 27 | 31 | 36 | 40 |
| | | | | | 3.914 | 3.962 | 4.011 | 14 | 5 | 10 | 14 | 19 | 24 | 29 | 34 | 38 | 43 |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ ← | Deg. | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ |

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End of tan-cot table for 1′ intervals on pages 44-47.

| A | 0′ | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ | 10′ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 76°00′ | 4.011 | 4.016 | 4.021 | 4.026 | 4.031 | 4.036 | 4.041 | 4.046 | 4.051 | 4.056 | 4.061 | 50′ |
| 10′ | 4.061 | 4.066 | 4.071 | 4.076 | 4.082 | 4.087 | 4.092 | 4.097 | 4.102 | 4.107 | 4.113 | 40′ |
| 20′ | 4.113 | 4.118 | 4.123 | 4.128 | 4.134 | 4.139 | 4.144 | 4.149 | 4.155 | 4.160 | 4.165 | 30′ |
| 30′ | 4.165 | 4.171 | 4.176 | 4.181 | 4.187 | 4.192 | 4.198 | 4.203 | 4.208 | 4.214 | 4.219 | 20′ |
| 40′ | 4.219 | 4.225 | 4.230 | 4.236 | 4.241 | 4.247 | 4.252 | 4.258 | 4.264 | 4.269 | 4.275 | 10′ |
| 50′ | 4.275 | 4.280 | 4.286 | 4.292 | 4.297 | 4.303 | 4.309 | 4.314 | 4.320 | 4.326 | 4.331 | 13°00′ |
| 77°00′ | 4.331 | 4.337 | 4.343 | 4.349 | 4.355 | 4.360 | 4.366 | 4.372 | 4.378 | 4.384 | 4.390 | 50′ |
| 10′ | 4.390 | 4.396 | 4.402 | 4.407 | 4.413 | 4.419 | 4.425 | 4.431 | 4.437 | 4.443 | 4.449 | 40′ |
| 20′ | 4.449 | 4.455 | 4.462 | 4.468 | 4.474 | 4.480 | 4.486 | 4.492 | 4.498 | 4.505 | 4.511 | 30′ |
| 30′ | 4.511 | 4.517 | 4.523 | 4.529 | 4.536 | 4.542 | 4.548 | 4.555 | 4.561 | 4.567 | 4.574 | 20′ |
| 40′ | 4.574 | 4.580 | 4.586 | 4.593 | 4.599 | 4.606 | 4.612 | 4.619 | 4.625 | 4.632 | 4.638 | 10′ |
| 50′ | 4.638 | 4.645 | 4.651 | 4.658 | 4.665 | 4.671 | 4.678 | 4.685 | 4.691 | 4.698 | 4.705 | 12°00′ |
| 78°00′ | 4.705 | 4.711 | 4.718 | 4.725 | 4.732 | 4.739 | 4.745 | 4.752 | 4.759 | 4.766 | 4.773 | 50′ |
| 10′ | 4.773 | 4.780 | 4.787 | 4.794 | 4.801 | 4.808 | 4.815 | 4.822 | 4.829 | 4.836 | 4.843 | 40′ |
| 20′ | 4.843 | 4.850 | 4.857 | 4.864 | 4.872 | 4.879 | 4.886 | 4.893 | 4.901 | 4.908 | 4.915 | 30′ |
| 30′ | 4.915 | 4.922 | 4.930 | 4.937 | 4.945 | 4.952 | 4.959 | 4.967 | 4.974 | 4.982 | 4.989 | 20′ |
| 40′ | 4.989 | 4.997 | 5.005 | 5.012 | 5.020 | 5.027 | 5.035 | 5.043 | 5.050 | 5.058 | 5.066 | 10′ |
| 50′ | 5.066 | 5.074 | 5.081 | 5.089 | 5.097 | 5.105 | 5.113 | 5.121 | 5.129 | 5.137 | 5.145 | 11°00′ |
| 79°00′ | 5.145 | 5.153 | 5.161 | 5.169 | 5.177 | 5.185 | 5.193 | 5.201 | 5.209 | 5.217 | 5.226 | 50′ |
| 10′ | 5.226 | 5.234 | 5.242 | 5.250 | 5.259 | 5.267 | 5.276 | 5.284 | 5.292 | 5.301 | 5.309 | 40′ |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20′ | 5.309 | 5.318 | 5.326 | 5.335 | 5.343 | 5.352 | 5.361 | 5.369 | 5.378 | 5.387 | 5.396 | 30′ |
| 30′ | 5.396 | 5.404 | 5.413 | 5.422 | 5.431 | 5.440 | 5.449 | 5.458 | 5.466 | 5.475 | 5.485 | 20′ |
| 40′ | 5.485 | 5.494 | 5.503 | 5.512 | 5.521 | 5.530 | 5.539 | 5.549 | 5.558 | 5.567 | 5.576 | 10′ |
| 50′ | 5.576 | 5.586 | 5.595 | 5.605 | 5.614 | 5.623 | 5.633 | 5.642 | 5.652 | 5.662 | 5.671 | 10°00′ |
| 80°00′ | 5.671 | 5.681 | 5.691 | 5.700 | 5.710 | 5.720 | 5:730 | 5.740 | 5.749 | 5.759 | 5.769 | 50′ |
| 10′ | 5.769 | 5.779 | 5.789 | 5.799 | 5.810 | 5.820 | 5.830 | 5.840 | 5.850 | 5.861 | 5.871 | 40′ |
| 20′ | 5.871 | 5.881 | 5.892 | 5.902 | 5.912 | 5.923 | 5.933 | 5.944 | 5.954 | 5.965 | 5.976 | 30′ |
| 30′ | 5.976 | 5.986 | 5.997 | 6.008 | 6.019 | 6.030 | 6.041 | 6.051 | 6.062 | 6.073 | 6.084 | 20′ |
| 40′ | 6.084 | 6.096 | 6.107 | 6.118 | 6.129 | 6.140 | 6.152 | 6.163 | 6.174 | 6.186 | 6.197 | 10′ |
| 50′ | 6.197 | 6.209 | 6.220 | 6.232 | 6.243 | 6.255 | 6.267 | 6.278 | 6.290 | 6.302 | 6.314 | 9°00′ |
| 81°00′ | 6.314 | 6.326 | 6.338 | 6.350 | 6.362 | 6.374 | 6.386 | 6.398 | 6.410 | 6.423 | 6.435 | 50′ |
| 10′ | 6.435 | 6.447 | 6.460 | 6.472 | 6.485 | 6.497 | 6.510 | 6.522 | 6.535 | 6.548 | 6.561 | 40′ |
| 20′ | 6.561 | 6.573 | 6.586 | 6.599 | 6.612 | 6.625 | 6.638 | 6.651 | 6.665 | 6.678 | 6.691 | 30′ |
| 30′ | 6.691 | 6.704 | 6.718 | 6.731 | 6.745 | 6.758 | 6.772 | 6.786 | 6.799 | 6.813 | 6.827 | 20′ |
| 40′ | 6.827 | 6 841 | 6.855 | 6.869 | 6.883 | 6.897 | 6.911 | 6.925 | 6.940 | 6.954 | 6.968 | 10′ |
| 50′ | 6.968 | 6.983 | 6.997 | 7.012 | 7.026 | 7.041 | 7.056 | 7.071 | 7.085 | 7.100 | 7.115 | 8°00′ |
| 82°00′ | 7.115 | 7.130 | 7.146 | 7.161 | 7.176 | 7.191 | 7.207 | 7.222 | 7.238 | 7.253 | 7.269 | 50′ |
| 10′ | 7.269 | 7.284 | 7.300 | 7.316 | 7.332 | 7.348 | 7.364 | 7.380 | 7.396 | 7.412 | 7.429 | 40′ |
| 20′ | 7.429 | 7.445 | 7.462 | 7.478 | 7.495 | 7.511 | 7.528 | 7.545 | 7.562 | 7.579 | 7.596 | 30′ |
| 30′ | 7.596 | 7.613 | 7.630 | 7.647 | 7.665 | 7.682 | 7.700 | 7.717 | 7.735 | 7.753 | 7.770 | 20′ |
| 40′ | 7.770 | 7.788 | 7.806 | 7.824 | 7.842 | 7.861 | 7.879 | 7.897 | 7.916 | 7.934 | 7.953 | 10′ |
| 50′ | 7.953 | 7.972 | 7.991 | 8.009 | 8.028 | 8.048 | 8.067 | 8.086 | 8.105 | 8.125 | 8.144 | 7°00′ |
| | 10′ | 9′ | 8′ | 7′ | 6′ | 5′ | 4′ | 3′ | 2′ | 1′ | 0′ | *A* |

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| A | 0′ | 1′ | 2′ | 3′ | 4′ | 5′ | 6′ | 7′ | 8′ | 9′ | 10′ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 83°00′ | 8.144 | 8.164 | 8.184 | 8.204 | 8.223 | 8.243 | 8.264 | 8.284 | 8.304 | 8.324 | 8.345 | 50′ |
| 10′ | 8.345 | 8.366 | 8.386 | 8.407 | 8.428 | 8.449 | 8.470 | 8.491 | 8.513 | 8.534 | 8.556 | 40′ |
| 20′ | 8.556 | 8.577 | 8.599 | 8.621 | 8.643 | 8.665 | 8.687 | 8.709 | 8.732 | 8.754 | 8.777 | 30′ |
| 30′ | 8.777 | 8.800 | 8.823 | 8.846 | 8.869 | 8.892 | 8.915 | 8.939 | 8.962 | 8.986 | 9.010 | 20′ |
| 40′ | 9.010 | 9.034 | 9.058 | 9.082 | 9.106 | 9.131 | 9.156 | 9.180 | 9.205 | 9.230 | 9.255 | 10′ |
| 50′ | 9.255 | 9.281 | 9.306 | 9.332 | 9.357 | 9.383 | 9.409 | 9.435 | 9.461 | 9.488 | 9.514 | 6°00′ |
| 84°00′ | 9.514 | 9.541 | 9.568 | 9.595 | 9.622 | 9.649 | 9.677 | 9.704 | 9.732 | 9.760 | 9.788 | 50′ |
| 10′ | 9.788 | 9.816 | 9.845 | 9.873 | 9.902 | 9 931 | 9.960 | 9.989 | 10.02 | 10 05 | 10.08 | 40′ |
| 20′ | 10.08 | 10.11 | 10.14 | 10.17 | 10.20 | 10.23 | 10.26 | 10.29 | 10.32 | 10.35 | 10 39 | 30′ |
| 30′ | 10.39 | 10 42 | 10.45 | 10.48 | 10.51 | 10.55 | 10.58 | 10.61 | 10.64 | 10.68 | 10.71 | 20′ |
| 40′ | 10.71 | 10.75 | 10.78 | 10.81 | 10.85 | 10.88 | 10.92 | 10.95 | 10.99 | 11 02 | 11.06 | 10′ |
| 50′ | 11.06 | 11.10 | 11.13 | 11 17 | 11.20 | 11.24 | 11.28 | 11.32 | 11.35 | 11.39 | 11.43 | 5°00′ |
| 85°00′ | 11.43 | 11.47 | 11.51 | 11.55 | 11.59 | 11.62 | 11.66 | 11.70 | 11.74 | 11.79 | 11.83 | 50′ |
| 10′ | 11.83 | 11 87 | 11.91 | 11.95 | 11 99 | 12.03 | 12.08 | 12.12 | 12.16 | 12.21 | 12.25 | 40′ |
| 20′ | 12.25 | 12.29 | 12.34 | 12.38 | 12.43 | 12.47 | 12.52 | 12.57 | 12.61 | 12.66 | 12.71 | 30′ |
| 30′ | 12.71 | 12.75 | 12.80 | 12.85 | 12.90 | 12.95 | 13.00 | 13.05 | 13 10 | 13.15 | 13 20 | 20′ |
| 40′ | 13.20 | 13.25 | 13.30 | 13.35 | 13.40 | 13.46 | 13.51 | 13.56 | 13.62 | 13.67 | 13.73 | 10′ |
| 50′ | 13.73 | 13 78 | 13.84 | 13.89 | 13.95 | 14 01 | 14.07 | 14.12 | 14.18 | 14.24 | 14.30 | 4°00′ |
| 86°00′ | 14.30 | 14.36 | 14.42 | 14.48 | 14 54 | 14.61 | 14.67 | 14.73 | 14.80 | 14.86 | 14.92 | 50′ |
| 10′ | 14.92 | 14.99 | 15.06 | 15 12 | 15.19 | 15 26 | 15.33 | 15.39 | 15.46 | 15 53 | 15.60 | 40′ |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20′ | 15.60 | 15.68 | 15.75 | 15.82 | 15.89 | 15.97 | 16.04 | 16.12 | 16.20 | 16.27 | 16.35 | 30′ |
| 30′ | 16.35 | 16.43 | 16.51 | 16.59 | 16.67 | 16.75 | 16.83 | 16.92 | 17.00 | 17.08 | 17.17 | 20′ |
| 40′ | 17.17 | 17.26 | 17.34 | 17.43 | 17.52 | 17.61 | 17.70 | 17.79 | 17.89 | 17.98 | 18.07 | 10′ |
| 50′ | 18.07 | 18.17 | 18.27 | 18.37 | 18.46 | 18.56 | 18.67 | 18.77 | 18.87 | 18.98 | 19.08 | 3°00′ |
| 87°00′ | 19.08 | 19.19 | 19.30 | 19.41 | 19 52 | 19.63 | 19.74 | 19.85 | 19.97 | 20.09 | 20.21 | 50′ |
| 10′ | 20.21 | 20.33 | 20.45 | 20.57 | 20.69 | 20.82 | 20.95 | 21.07 | 21.20 | 21.34 | 21.47 | 40′ |
| 20′ | 21.47 | 21.61 | 21.74 | 21.88 | 22.02 | 22.16 | 22.31 | 22.45 | 22.60 | 22.75 | 22.90 | 30′ |
| 30′ | 22.90 | 23.06 | 23.21 | 23.37 | 23.53 | 23.69 | 23.86 | 24.03 | 24 20 | 24 37 | 24.54 | 20′ |
| 40′ | 24.54 | 24.72 | 24.90 | 25.08 | 25.26 | 25.45 | 25.64 | 25.83 | 26.03 | 26.23 | 26.43 | 10′ |
| 50′ | 26.43 | 26.64 | 26.84 | 27.06 | 27.27 | 27.49 | 27.71 | 27.94 | 28.17 | 28.40 | 28.64 | 2°00′ |
| 88°00′ | 28.64 | 28.88 | 29.12 | 29.37 | 29.62 | 29.88 | 30.14 | 30.41 | 30.68 | 30.96 | 31.24 | 50′ |
| 10′ | 31.24 | 31.53 | 31.82 | 32.12 | 32.42 | 32.73 | 33.05 | 33.37 | 33.69 | 34.03 | 34.37 | 40′ |
| 20′ | 34.37 | 34.72 | 35.07 | 35.43 | 35.80 | 36.18 | 36.56 | 36.96 | 37.36 | 37.77 | 38.19 | 30′ |
| 30′ | 38.19 | 38.62 | 39.06 | 39.51 | 39.97 | 40.44 | 40.92 | 41.41 | 41.92 | 42.43 | 42.96 | 20′ |
| 40′ | 42.96 | 43.51 | 44.07 | 44.64 | 45.23 | 45.83 | 46.45 | 47.09 | 47.74 | 48.41 | 49.10 | 10′ |
| 50′ | 49.10 | 49.82 | 50 55 | 51.30 | 52.08 | 52.88 | 53.71 | 54.56 | 55.44 | 56.35 | 57.29 | 1°00′ |
| 89°00′ | 57.29 | 58.26 | 59 27 | 60.31 | 61.38 | 62.50 | 63.66 | 64.86 | 66.11 | 67.40 | 68.75 | 50′ |
| 10′ | 68.75 | 70.15 | 71.62 | 73.14 | 74.73 | 76.39 | 78.13 | 79 94 | 81.85 | 83.84 | 85.94 | 40′ |
| 20′ | 85.94 | 88.14 | 90·46 | 92 91 | 95.49 | 98.22 | 101·1 | 104.2 | 107.4 | 110.9 | 114.6 | 30′ |
| 30′ | 114.6 | 118.5 | 122 8 | 127.3 | 132.2 | 137.5 | 143.2 | 149.5 | 156.3 | 163.7 | 171.9 | 20′ |
| 40′ | 171.9 | 180.9 | 191.0 | 202.2 | 214.9 | 229.2 | 245.6 | 264.4 | 286.5 | 312.5 | 343.8 | 10′ |
| 50′ | 343.8 | 382.0 | 429.7 | 491.1 | 573.0 | 687.5 | 859 4 | 1146 | 1719 | 3438 | ∞ | 0°00′ |
| | 10′ | 9′ | 8′ | 7′ | 6′ | 5′ | 4′ | 3′ | 2′ | 1′ | 0′ | A |

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## 8. Conversion of Degrees to Radians (see Sec. 181)

Arc lengths of a circle of radius 1

| Deg. | Radians (arc) | Deg. | Radians (arc) | Deg. | Radians (arc) | Min. | Radians (arc) | Min. | Radians (arc) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0000 | 35 | 0.6109 | 70 | 1.2217 | 0 | 0.0000 | 30 | 0.0087 |
| 1 | 0.0175 | 36 | 0.6283 | 71 | 1.2392 | 1 | 0.0003 | 31 | 0.0090 |
| 2 | 0.0349 | 37 | 0.6458 | 72 | 1.2566 | 2 | 0.0006 | 32 | 0.0093 |
| 3 | 0.0524 | 38 | 0.6632 | 73 | 1.2741 | 3 | 0.0009 | 33 | 0.0096 |
| 4 | 0.0698 | 39 | 0.6807 | 74 | 1.2915 | 4 | 0.0012 | 34 | 0.0099 |
| 5 | 0.0873 | 40 | 0.6981 | 75 | 1.3090 | 5 | 0.0015 | 35 | 0.0102 |
| 6 | 0.1047 | 41 | 0.7156 | 76 | 1.3265 | 6 | 0.0017 | 36 | 0.0105 |
| 7 | 0.1222 | 42 | 0.7330 | 77 | 1.3439 | 7 | 0.0020 | 37 | 0.0108 |
| 8 | 0.1396 | 43 | 0.7505 | 78 | 1.3614 | 8 | 0.0023 | 38 | 0.0111 |
| 9 | 0.1571 | 44 | 0.7679 | 79 | 1.3788 | 9 | 0.0026 | 39 | 0.0113 |
| 10 | 0.1745 | 45 | 0.7854 | 80 | 1.3963 | 10 | 0.0029 | 40 | 0.0116 |
| 11 | 0.1920 | 46 | 0.8029 | 81 | 1.4137 | 11 | 0.0032 | 41 | 0.0119 |
| 12 | 0.2094 | 47 | 0.8203 | 82 | 1.4312 | 12 | 0.0035 | 42 | 0.0122 |
| 13 | 0.2269 | 48 | 0.8378 | 83 | 1.4486 | 13 | 0.0038 | 13 | 0.0125 |
| 14 | 0.2443 | 49 | 0.8552 | 84 | 1.4661 | 14 | 0.0041 | 44 | 0.0128 |
| 15 | 0.2618 | 50 | 0.8727 | 85 | 1.4835 | 15 | 0.0044 | 45 | 0.0131 |
| 16 | 0.2793 | 51 | 0.8901 | 86 | 1.5010 | 16 | 0.0047 | 46 | 0.0134 |
| 17 | 0.2967 | 52 | 0.9076 | 87 | 1.5184 | 17 | 0.0049 | 47 | 0.0137 |
| 18 | 0.3142 | 53 | 0.9250 | 88 | 1.5359 | 18 | 0.0052 | 48 | 0.0140 |
| 19 | 0.3316 | 54 | 0.9425 | 89 | 1.5533 | 19 | 0.0055 | 49 | 0.0143 |
| 20 | 0 3491 | 55 | 0.9599 | 90 | 1.5708 | 20 | 0.0058 | 50 | 0.0145 |
| 21 | 0.3665 | 56 | 0.9774 | 91 | 1.5882 | 21 | 0 0061 | 51 | 0.0148 |
| 22 | 0.3840 | 57 | 0.9948 | 92 | 1.6057 | 22 | 0.0064 | 52 | 0.0151 |
| 23 | 0.4014 | 58 | 1.0123 | 93 | 1 6232 | 23 | 0.0067 | 53 | 0.0154 |
| 24 | 0.4189 | 59 | 1.0297 | 94 | 1.6406 | 24 | 0.0070 | 54 | 0.0157 |
| 25 | 0.4363 | 60 | 1.0472 | 95 | 1.6581 | 25 | 0 0073 | 55 | 0.0160 |
| 26 | 0.4538 | 61 | 1.0647 | 96 | 1.6755 | 26 | 0.0076 | 56 | 0.0163 |
| 27 | 0.4712 | 62 | 1.0821 | 97 | 1.6930 | 27 | 0.0079 | 57 | 0.0166 |
| 28 | 0.4887 | 63 | 1.0996 | 98 | 1.7104 | 28 | 0.0081 | 58 | 0.0169 |
| 29 | 0.5061 | 64 | 1.1170 | 99 | 1.7279 | 29 | 0.0084 | 59 | 0.0172 |
| 30 | 0.5236 | 65 | 1.1345 | 100 | 1.7453 | | | | |
| 31 | 0.5411 | 66 | 1.1519 | 180 | 3.1416 | | | | |
| 32 | 0 5585 | 67 | 1.1694 | 200 | 3.4907 | | | | |
| 33 | 0.5760 | 68 | 1.1868 | 300 | 5.2360 | | | | |
| 34 | 0.5934 | 69 | 1.2043 | 360 | 6.2832 | | | | |

## 9. Conversion of Radians to Degrees and Minutes (see Sec. 181)

| Radi-ans | Deg. and min. | Radi-ans | Deg. and min. | Radi-ans | Deg. and min. | Radi-ans | Min. | Radi-ans | Min. |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 57°18′ | 0.1 | 5°44′ | 0.01 | 0°34′ | 0.001 | 0°03′ | 0.0001 | 0°00′ |
| 2 | 114°35′ | 0.2 | 11°28′ | 0.02 | 1°09′ | 0.002 | 0°07′ | 0 0002 | 0°01′ |
| 3 | 171°53′ | 0.3 | 17°11′ | 0.03 | 1°43′ | 0.003 | 0°10′ | 0.0003 | 0°01′ |
| 4 | 229°11′ | 0.4 | 22°55′ | 0.04 | 2°18′ | 0.004 | 0°14′ | 0.0004 | 0°01′ |
| 5 | 286°29′ | 0 5 | 28°39′ | 0.05 | 2°52′ | 0.005 | 0°17′ | 0.0005 | 0°02′ |
| 6 | 343°46′ | 0.6 | 34°23′ | 0.06 | 3°26′ | 0.006 | 0°21′ | 0.0006 | 0°02′ |
| 7 | 401°04′ | 0.7 | 40°06′ | 0.07 | 4°01′ | 0.007 | 0°24′ | 0.0007 | 0°02′ |
| 8 | 458°22′ | 0.8 | 45°50′ | 0.08 | 4°35′ | 0.008 | 0°28′ | 0.0008 | 0°03′ |
| 9 | 515°40′ | 0.9 | 51°34′ | 0.09 | 5°09′ | 0.009 | 0°31′ | 0.0009 | 0°03′ |

## 10. Table of Primes $< 6000$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 193 | 449 | 733 | 1031 | 1321 | 1637 | 1997 | 2333 |
| 3 | 197 | 457 | 739 | 1033 | 1327 | 1657 | 1999 | 2339 |
| 5 | 199 | 461 | 743 | 1039 | 1361 | 1663 | 2003 | 2341 |
| 7 | 211 | 463 | 751 | 1049 | 1367 | 1667 | 2011 | 2347 |
| 11 | 223 | 467 | 757 | 1051 | 1373 | 1669 | 2017 | 2351 |
| 13 | 227 | 479 | 761 | 1061 | 1381 | 1693 | 2027 | 2357 |
| 17 | 229 | 487 | 769 | 1063 | 1399 | 1697 | 2029 | 2371 |
| 19 | 233 | 491 | 773 | 1069 | 1409 | 1699 | 2039 | 2377 |
| 23 | 239 | 499 | 787 | 1087 | 1423 | 1709 | 2053 | 2381 |
| 29 | 241 | 503 | 797 | 1091 | 1427 | 1721 | 2063 | 2383 |
| 31 | 251 | 509 | 809 | 1093 | 1429 | 1723 | 2069 | 2389 |
| 37 | 257 | 521 | 811 | 1097 | 1433 | 1733 | 2081 | 2393 |
| 41 | 263 | 523 | 821 | 1103 | 1439 | 1741 | 2083 | 2399 |
| 43 | 269 | 541 | 823 | 1109 | 1447 | 1747 | 2087 | 2411 |
| 47 | 271 | 547 | 827 | 1117 | 1451 | 1753 | 2089 | 2417 |
| 53 | 277 | 557 | 829 | 1123 | 1453 | 1759 | 2099 | 2423 |
| 59 | 281 | 563 | 839 | 1129 | 1459 | 1777 | 2111 | 2437 |
| 61 | 283 | 569 | 853 | 1151 | 1471 | 1783 | 2113 | 2441 |
| 67 | 293 | 571 | 857 | 1153 | 1481 | 1787 | 2129 | 2447 |
| 71 | 307 | 577 | 859 | 1163 | 1483 | 1789 | 2131 | 2459 |
| 73 | 311 | 587 | 863 | 1171 | 1487 | 1801 | 2137 | 2467 |
| 79 | 313 | 593 | 877 | 1181 | 1489 | 1811 | 2141 | 2473 |
| 83 | 317 | 599 | 881 | 1187 | 1493 | 1823 | 2143 | 2477 |
| 89 | 331 | 601 | 883 | 1193 | 1499 | 1831 | 2153 | 2503 |
| 97 | 337 | 607 | 887 | 1201 | 1511 | 1847 | 2161 | 2521 |
| 101 | 347 | 613 | 907 | 1213 | 1523 | 1861 | 2179 | 2531 |
| 103 | 349 | 617 | 911 | 1217 | 1531 | 1867 | 2203 | 2539 |
| 107 | 353 | 619 | 919 | 1223 | 1543 | 1871 | 2207 | 2543 |
| 109 | 359 | 631 | 929 | 1229 | 1549 | 1873 | 2213 | 2549 |
| 113 | 367 | 641 | 937 | 1231 | 1553 | 1877 | 2221 | 2551 |
| 127 | 373 | 643 | 941 | 1237 | 1559 | 1879 | 2237 | 2557 |
| 131 | 379 | 647 | 947 | 1249 | 1567 | 1889 | 2239 | 2579 |
| 137 | 383 | 653 | 953 | 1259 | 1571 | 1901 | 2243 | 2591 |
| 139 | 389 | 659 | 967 | 1277 | 1579 | 1907 | 2251 | 2593 |
| 149 | 397 | 661 | 971 | 1279 | 1583 | 1913 | 2267 | 2609 |
| 151 | 401 | 673 | 977 | 1283 | 1597 | 1931 | 2269 | 2617 |
| 157 | 409 | 677 | 983 | 1289 | 1601 | 1933 | 2273 | 2621 |
| 163 | 419 | 683 | 991 | 1291 | 1607 | 1949 | 2281 | 2633 |
| 167 | 421 | 691 | 997 | 1297 | 1609 | 1951 | 2287 | 2647 |
| 173 | 431 | 701 | 1009 | 1301 | 1613 | 1973 | 2293 | 2657 |
| 179 | 433 | 709 | 1013 | 1303 | 1619 | 1979 | 2297 | 2659 |
| 181 | 439 | 719 | 1019 | 1307 | 1621 | 1987 | 2309 | 2663 |
| 191 | 443 | 727 | 1021 | 1319 | 1627 | 1993 | 2311 | 2671 |

*Continued*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2677 | 3011 | 3373 | 3727 | 4093 | 4481 | 4871 | 5233 | 5639 |
| 2683 | 3019 | 3389 | 3733 | 4099 | 4483 | 4877 | 5237 | 5641 |
| 2687 | 3023 | 3391 | 3739 | 4111 | 4493 | 4889 | 5261 | 5647 |
| 2689 | 3037 | 3407 | 3761 | 4127 | 4507 | 4903 | 5273 | 5651 |
| 2693 | 3041 | 3413 | 3767 | 4129 | 4513 | 4909 | 5279 | 5653 |
| 2699 | 3049 | 3433 | 3769 | 4133 | 4517 | 4919 | 5281 | 5667 |
| 2707 | 3061 | 3449 | 3779 | 4139 | 4519 | 4931 | 5297 | 5659 |
| 2711 | 3067 | 3457 | 3793 | 4153 | 4523 | 4933 | 5303 | 5669 |
| 2713 | 3079 | 3461 | 3797 | 4157 | 4547 | 4937 | 5309 | 5683 |
| 2719 | 3083 | 3463 | 3803 | 4159 | 4549 | 4943 | 5323 | 5689 |
| 2729 | 3089 | 3467 | 3821 | 4177 | 4561 | 4951 | 5333 | 5693 |
| 2731 | 3109 | 3469 | 3823 | 4201 | 4567 | 4957 | 5347 | 5701 |
| 2741 | 3119 | 3491 | 3833 | 4211 | 4583 | 4967 | 5351 | 5711 |
| 2749 | 3121 | 3499 | 3847 | 4217 | 4591 | 4969 | 5381 | 5717 |
| 2753 | 3137 | 3511 | 3851 | 4219 | 4597 | 4973 | 5387 | 5737 |
| 2767 | 3163 | 3517 | 3853 | 4229 | 4603 | 4987 | 5393 | 5741 |
| 2777 | 3167 | 3527 | 3863 | 4231 | 4621 | 4993 | 5399 | 5743 |
| 2789 | 3169 | 3529 | 3877 | 4241 | 4637 | 4999 | 5407 | 5749 |
| 2791 | 3181 | 3533 | 3881 | 4243 | 4639 | 5003 | 5413 | 5779 |
| 2797 | 3187 | 3539 | 3889 | 4253 | 4643 | 5009 | 5417 | 5783 |
| 2801 | 3191 | 3541 | 3907 | 4259 | 4649 | 5011 | 5419 | 5791 |
| 2803 | 3203 | 3547 | 3911 | 4261 | 4651 | 5021 | 5431 | 5801 |
| 2819 | 3219 | 3557 | 3917 | 4271 | 4657 | 5023 | 5437 | 5807 |
| 2833 | 3217 | 3559 | 3919 | 4273 | 4663 | 5039 | 5441 | 5813 |
| 2837 | 3221 | 3571 | 3923 | 4283 | 4673 | 5051 | 5443 | 5821 |
| 2843 | 3229 | 3581 | 3929 | 4289 | 4679 | 5059 | 5449 | 5827 |
| 2851 | 3251 | 3583 | 3931 | 4297 | 4691 | 5077 | 5471 | 5839 |
| 2857 | 3253 | 3593 | 3943 | 4327 | 4703 | 5081 | 5477 | 5843 |
| 2861 | 3257 | 3607 | 3947 | 4337 | 4721 | 5087 | 5479 | 5849 |
| 2879 | 3259 | 3613 | 3967 | 4339 | 4723 | 5099 | 5483 | 5851 |
| 2887 | 3271 | 3617 | 3989 | 4349 | 4729 | 5101 | 5501 | 5857 |
| 2897 | 3299 | 3623 | 4001 | 4357 | 4733 | 5107 | 5503 | 5861 |
| 2903 | 3301 | 3631 | 4003 | 4363 | 4751 | 5113 | 5507 | 5867 |
| 2909 | 3307 | 3637 | 4007 | 4373 | 4759 | 5119 | 5519 | 5869 |
| 2917 | 3313 | 3643 | 4013 | 4391 | 4783 | 5147 | 5521 | 5879 |
| 2927 | 3319 | 3659 | 4019 | 4397 | 4787 | 5153 | 5527 | 5881 |
| 2939 | 3323 | 3671 | 4021 | 4409 | 4789 | 5167 | 5531 | 5897 |
| 2953 | 3329 | 3673 | 4027 | 4421 | 4793 | 5171 | 5557 | 5903 |
| 2957 | 3331 | 3677 | 4049 | 4423 | 4799 | 5179 | 5563 | 5923 |
| 2963 | 3343 | 3691 | 4051 | 4441 | 4801 | 5189 | 5569 | 5927 |
| 2969 | 3347 | 3697 | 4057 | 4447 | 4813 | 5197 | 5573 | 5939 |
| 2971 | 3359 | 3701 | 4073 | 4451 | 4817 | 5209 | 5581 | 5953 |
| 2999 | 3361 | 3709 | 4079 | 4457 | 4831 | 5227 | 5591 | 5981 |
| 3001 | 3371 | 3719 | 4091 | 4463 | 4861 | 5231 | 5623 | 5987 |

## 11. Mathematical Symbols

| *Symbol* | *Meaning* | *Example* |
|---|---|---|
| $=$ | is equal to | $a=b$ |
| $\neq$ | is not equal to | $a\neq b$ |
| $\approx$ | is approximately equal to | $a\approx b$ |
| $>$ | is greater than | $5>2$ |
| $<$ | is less than | $3<10$ |
| $\geqslant$ | is greater than or equal to | $a\geqslant b$ |
| $\leqslant$ | is less than or equal to | $a\leqslant b$ |
| $\lvert\ \rvert$ | absolute value | $\lvert a\rvert$ |
| $\sqrt[n]{\ }$ | $n$th root of (usually means the principal $n$th root) | $\sqrt[3]{8}=2$ |
| ! | factorial | $5!=1\cdot 2\cdot 3\cdot 4\cdot 5=120$ (read: factorial 5, or 5 factorial) |
| $\log_b$ | logarithm to the base $b$ | $\log_2 8=3$ |
| $\log_{10}$ (or log) | common logarithm ($\log a$ is used for $\log_{10} a$ when the context shows that the base is 10) | $\log 100=2$ |
| ln (or $\log_e$) | natural (Napierian) logarithm (to the base $e$) | |
| lim | limit | |
| const | constant | |
| $\Sigma$ | summation of | |
| $\triangle$ | triangle | $\triangle ABC$ (pl. $\triangle$s $ABC$ and $DEF$) |
| $\angle$ | angle | $\angle ABC$ |
| $\overset{\frown}{\ }$ | arc | $\overset{\frown}{AB}$ |
| $\parallel$ | is parallel to | $AB\parallel CD$ |
| $\perp$ | is perpendicular to | $AB\perp CD$ |
| $\sim$ | is similar to | $\triangle ABC\sim\triangle DEF$ |
| $\pi$ | ratio of the circumference of a circle to the diameter, equal to 3.1415926536... | |
| $^\circ$ | degree | $10^\circ\ 30'\ 35''$ |
| $'$ | minute | |
| $''$ | second | |
| sin | sine | $\sin 30^\circ=\frac{1}{2}$ |
| cos | cosine | $\cos\frac{\pi}{2}=0$ |
| tan | tangent | $\tan 40^\circ=0.8391$ |
| cot | cotangent | $\cot 25^\circ 10'=2.128$ |
| sec | secant | $\sec 60^\circ=2$ |
| csc | cosecant | $\csc 90^\circ=1$ |
| arcsin | arc sine | $\arcsin\frac{1}{2}=30^\circ$ |
| arccos | arc cosine | $\arccos 0=\frac{\pi}{2}$ |
| arctan | arc tangent | $\arctan 0.8391=40^\circ$ |
| arccot | arc cotangent | $\operatorname{arccot} 2.128=25^\circ 10'$ |
| arcsec | arc secant | $\operatorname{arcsec} 2=60^\circ$ |
| arccsc | arc cosecant | $\operatorname{arccsc} 1=90^\circ$ |

## 12. The Metric System of Measurement

*Linear Measure*
1 kilometre (km) = 1000 metres (m)
1 metre (m) = 10 decimetres (dm) = 100 centimetres (cm)
1 decimetre (dm) = 10 centimetres (cm)
1 centimetre (cm) = 10 millimetres (mm)

*Square Measure (area)*

1 square kilometre ($km^2$) = 1,000,000 square metres ($m^2$)
1 square metre ($m^2$) = 100 square decimetres ($dm^2$) = 10,000 square centimetres ($cm^2$)
1 hectare (ha) = 100 ares = 10,000 square metres ($m^2$)
1 are = 100 square metres ($m^2$)

*Cubic Measure (volume)*

1 cubic metre ($m^3$) = 1000 cubic decimetres ($dm^3$) = 1,000,000 cubic centimetres ($cm^3$)
1 cubic decimetre ($dm^3$) = 1000 cubic centimetres ($cm^3$)
1 litre (l) = 1 cubic decimetre ($dm^3$)
1 hectolitre (hl) = 100 litres (l)

*Metric Weight*

1 ton (metric) = 1000 kilograms (kg)
1 centner = 100 kilograms (kg)
1 kilogram (kg) = 1000 grams (g)
1 gram (g) = 1000 milligrams (mg)

## 13. Some Old Russian Measures

*Linear Measure*
1 versta = 500 sagenes = 1500 arshins = 3500 feet = 1066.8 m
1 sagene = 3 arshins = 48 verchoks = 7 feet = 84 inches = 2.1336 m
1 arshin = 16 verchoks = 71.12 cm
1 verchok = 4.450 cm
1 foot = 12 inches = 0.3048 m
1 inch = 2.540 cm
1 nautical mile = 1852.2 m in USSR, 1853.18 m in Britain, 1853.25 m in USA

*Weight*
1 pood = 40 pounds = 16.380 kg
1 pound = 0.40951 kg

## 14. The Greek Alphabet

| | | | |
|---|---|---|---|
| Αα | alpha | Νν | nu |
| Ββ | beta | Ξξ | xi |
| Γγ | gamma | Οο | omicron |
| Δδ | delta | Ππ | pi |
| Εε | epsilon | Ρρ | rho |
| Ζζ | zeta | Σσ | sigma |
| Ηη | eta | Ττ | tau |
| Θθ | theta | Υυ | upsilon |
| Ιι | iota | Φφ | phi |
| Κκ | kappa | Χχ | chi |
| Λλ | lambda | Ψψ | psi |
| Μμ | mu | Ωω | omega |

# ARITHMETIC

### 15. The Subject of Arithmetic

*Arithmetic* is the science of numbers, the name stemming from the Greek word "arithmos" which means "number". It involves the most elementary properties of numbers and rules of calculation. Deeper properties of numbers are studied in the theory of numbers.

### 16. Whole Numbers (Natural Numbers)

The first conceptions of number were acquired by man in remote antiquity (see Sec. 17). It began with the counting of people, animals, and the various articles and possessions of primitive man. Counting produced the numbers one, two, three, etc., which are now called *natural numbers*. In arithmetic they are also referred to as *whole numbers* or integers (the term "integer" has a broader meaning in mathematics; see Sec. 67).

The concept of a natural number is one of the most elementary notions. The only way to explain it is by demonstration. In the third century B. C., Euclid defined number (natural number) as a "collection made up of units"; similar definitions appear in textbooks even today. But the words "collection", or "group", or "aggregate", etc. do not seem to be any more comprehensible than the word "number".

The sequence of whole numbers

$$1, 2, 3, 4, 5, \ldots$$

goes on without end and is called the *set of natural numbers*.

### 17. The Limits of Counting

In primitive society, man could hardly count at all. He was able to distinguish groups of two and three objects, anything beyond that being thought of as "many". This was obviously not counting, it was only a beginning.

Gradually larger groups were distinguished, giving rise to the notions of "four", "five", "six", "seven". For a long time, the word "seven" was used in some languages to denote an indefinitely large quantity.

As man's activities became more intricate, the counting process developed and gave rise to a variety of reckoning devices: the making of notches on sticks and trees, knots in ropes, groups of stones, etc.*

The human hand with its five fingers was an invaluable natural tool for counting. It could not preserve the information it conveyed but it was ready at hand, so to speak, and very mobile. The language of primitive man was poor; gesticulations often made up for lack of words, and numbers (for which there were no names) were demonstrated in finger counting (this is even done nowadays if two persons speaking different languages do not understand each other).

It is quite natural that the newly originating names for "large" numbers were often constructed on the basis of the number 10 corresponding to the 10 fingers. Certain peoples developed a number system based on 5, the fingers of one hand, or on 20 which is the total number of fingers and toes (see Sec. 18).

During the early stages of man's development, the range of numbers expanded very slowly. Counting proceeded through the first tens and only much later reached one hundred. In many languages, the number 40 represented the limit and designated an indefinitely large quantity.

When the counting process reached ten tens and a name was given to the number 100, it was also used to denote an indefinitely large number (in some languages, Tartar for instance, one and the same word is used to denote 40 and 100). The very same process occurred again with the numbers thousand, ten thousand, and million.

### 18. The Decimal System of Numeration

In many modern languages, the names of all numbers up to a million are made up of 37 words denoting the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,

* Counting by means of stones (pebbles) served as the starting point for more sophisticated counting devices such as, for example, the Russian abacus, the Chinese abacus (or suan phan), the ancient Egyptian abacus (a board divided into strips where counters were placed). Other peoples had similar devices. In Latin, the idea of counting is expressed by the word "calculatio" (whence the English word "calculation") coming from "calculus", which means "pebble".

19, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000 (for example, 918,742: nine hundred eighteen thousand seven hundred forty-two). In turn, the names of these 37 numbers are, as a rule, built up from the names of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10, 100, 1000. The underlying element of all these word formations is the number 10; that is why the modern system of numbers is called the *decimal system of numeration*. The exceptional role of 10 is due to the fact that we have 10 fingers on our two hands (see Sec. 17).

That is the general rule, but a wide variety of exceptions are evident in various languages. These are due to historical peculiarities in the development of the counting process.

In the Turkic languages (Azerbaijan, Uzbek, Turkmen, Kazakh, Tartar, Turkish, etc.) the exceptions are the names of the numbers 20, 30, 40, 50, whereas 60, 70, 80, 90 are formed on the basis of the names for 6, 7, 8 and 9. In Mongolian, on the contrary, the names of the numbers 20, 30, 40, 50 follow the general rule, while 60, 70, 80 and 90 are exceptions. In Russian there is one exception, the name for "forty". In French the names for the numbers 20 and 80 retain the nondecimal names, 80 is quatrevingt (four twenties). This is a remnant of the ancient vigesimal system of numbers (based on 20—the total number of fingers and toes). In Latin also the name for 20 is nondecimal (viginti), but that for 80 (octoginta) is decimal and comes from 8 (octo). On the other hand, the names for the numbers 18 and 19 are built up from 20 by subtraction: 20—2 and 20—1 (duodeviginti, undeviginti, that is to say, two from twenty and one from twenty). The names of the numbers 200, 300, 400, 500, 600, 700, 800, and 900 in all modern languages are constructed on the decimal (scale-10) basis.

### 19. Development of the Number Concept

In the counting process, unity is the smallest number. There is no need to subdivide it, nor is it even possible at times (adding half a stone to two stones yields three stones, not $2^1/_2$, and of course it is impossible to select a committee made up of $2^1/_2$ persons). However, unity often has to be broken up into parts when measuring lengths by means of steps ($2^1/_2$ steps long, and the like). For this reason, the notion of a *fractional number* (see Secs. 30 and 45) was known in remote antiquity. Subsequent development

saw the expansion of the concept of number to *irrational numbers* (Sec. 91), *negative numbers* (Sec. 67), and *complex numbers* (Secs. 92 and 98).

Zero (null) was a long time in entering the family of numbers. At first, zero (null) had the meaning of absence of any number (the Latin "nullum" literally means "nothing"). If say 3 is taken away from 3 we have nothing. For that "nothing" to be considered a number, we had to wait for negative numbers to appear (see Sec. 67).

### 20. Numerals

*A numeral* is a written sign depicting a number. In the most ancient times, numbers were denoted by straight-line strokes ("rods"): one rod depicted unity, two rods, a two, and so forth. This notation originated from the use of notches. It still exists in the Roman numerals (see Sec. 21, Item 5) which denote the numbers 1, 2, 3.

This notation is inconvenient for writing large numbers and so special symbols were used to depict the number 10 (in accordance with the decimal system of numeration, see Sec. 18) and, in some languages, the number 5 as well (in accordance with quinary numeration which is based on the number of fingers of the human hand). Later, symbols were invented for still larger numbers. These symbols exhibited a variety of forms in the different languages and underwent considerable modifications in the course of time. There was also considerable variety in the *systems of numeration*, that is, modes of combining digits to form large numbers. However, in most number systems the 10-scale was pre-eminent and formed the basis of the decimal system of numeration (see Sec. 18).

### 21. Systems of Numeration

**1. Ancient Greek numeration.** The so-called *Attic system* of numeration was used in ancient Greece. The numbers 1, 2, 3, 4 were denoted by vertical strokes **I, II, III, IIII.** The number 5 had the symbol **Γ** (the first letter "pi", in its ancient form, of the word "pente", five); the numbers 6, 7,

8, 9 were written as $\Gamma\mathrm{I}, \Gamma\mathrm{II}, \Gamma\mathrm{III}, \Gamma\mathrm{IIII}$. The number 10 was depicted as $\Delta$ (the first letter of "deca", ten). The numbers 100, 1000, and 10,000 were denoted by $\mathrm{H}, \mathrm{X}, \mathrm{M}$ the initial letters of the corresponding words. The numbers 50, 500, and 5000 were given as combinations of the signs for 5 and 10, 5 and 100, 5 and 1000, namely: $\overline{\Gamma}, \overline{\mathrm{H}}, \overline{\mathrm{X}}$. The remaining numbers, within the first ten thousand, were written as follows:

$$\mathrm{HH}\overline{\Gamma}\Gamma\mathrm{I}=256, \quad \mathrm{XX}\overline{\Gamma}\mathrm{I}=2051,$$

$$\mathrm{HHH}\overline{\Gamma}\Delta\Delta\Delta\mathrm{II}=382, \quad \overline{\Gamma}\mathrm{XX}\overline{\Gamma}\mathrm{HHH}=7800$$

and so forth.

In the third century B. C., the Attic numeration gave way to the so-called *Ionian system*. Here, the numbers from 1 to 9 were denoted by the first nine letters of the alphabet (the letter $\digamma$, vau, $\varsigma$, koppa, and $\sampi$ sampi, are archaic; the Greek alphabet is given in Sec. 14):

$$\alpha=1, \quad \beta=2, \quad \gamma=3, \quad \delta=4, \quad \varepsilon=5, \quad \varsigma=6, \quad \zeta=7,$$
$$\eta=8, \quad \vartheta=9$$

the numbers from 10 to 90, by the next nine letters:

$$\iota=10, \quad \varkappa=20, \quad \lambda=30, \quad \mu=40, \quad \nu=50, \quad \xi=60,$$
$$\mathrm{o}=70, \quad \pi=80, \quad \varsigma=90$$

the numbers from 100 to 900, by the last nine letters:

$$\rho=100, \quad \sigma=200, \quad \tau=300, \quad \upsilon=400, \quad \varphi=500;$$
$$\chi=600, \quad \psi=700, \quad \omega=800, \quad \sampi=900$$

Thousands and tens of thousands were denoted by the same numerals preceded by a stroke or accent:

$$'\alpha=1000, \quad '\beta=2000, \quad \text{etc.}$$

A bar was placed over numerals in order to distinguish them from letters. For example,

$$\overline{\iota\eta}=18; \quad \overline{\mu\zeta}=47; \quad \overline{\upsilon\zeta}=407; \quad \overline{\chi\varkappa\alpha}=621, \quad \overline{\chi\varkappa}=620, \text{ etc.}$$

In ancient times, the same alphabetic numeration was used by the Jews, Arabs, and many other peoples of the Near East. It is not known what people used them first.

**2. Slavic numeration.** The Slavic peoples of the south and east of Europe used the alphabetic system of notation for writing numbers. In some cases, the numerical values of the letters were established in the order of the Slavic alphabet, in others (including Russia) not all letters were used as numerals but only those found in the Greek alphabet. The letter used as a numeral was surmounted by a special symbol (see accompanying table), and the numerical values of the letters increased in the order of the letters in the Greek alphabet (the sequence of letters in the Slavic alphabet was somewhat different).

This Slavic numeration persisted in Russia till the end of the 17th century. Under Peter the First, the dominant system of numeration was the Arabic (see Item 6 below) which is still in use today. The Slavic numeration persists, however, in clerical works.

**The Slavic Numerals**

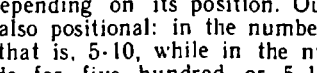
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| А | В | Г | Д | Є | Ѕ | З | И | Ѳ |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| І | К | Л | М | Н | Ѯ | О | П | Ч |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
| Р | С | Т | Ѵ | Ф | Х | Ѱ | Ѡ | Ц |
| 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 | 900 |

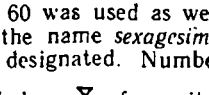
**3. The Ancient Armenian and Georgian systems of numeration.** Both Armenians and Georgians used the alphabetic principle of numeration. But the ancient alphabets of these peoples had far more letters than did the Greeks. This enab-

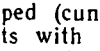
led them to use special symbols for the numbers 1000, 2000, 3000, 4000, 5000, 6000, 7000, 8000, and 9000. The numerical values of the letters followed the order of the letters in the alphabets of these peoples.

The alphabetic numeration persisted till the 18th century although the Arabic numeration was used occasionally much earlier (in the Georgian literature such instances go back to the 10 or 11th century; sources of Armenian mathematical literature reveal such usage no earlier than the 15th century). In Armenia the alphabetic numeration is still used in designations of chapters, stanzas, and the like. In Georgia the alphabetic numeration has gone out of use altogether.

**4. Babylonian Positional System of Numeration.** Approximately 40 centuries prior to the Christian era, the ancient Babylonians developed a positional system of notation for their numeration. This is a mode of representing numbers in which one and the same digit is capable of denoting different numbers depending on its position. Our present-day numeration is also positional: in the number 52, the digit 5 denotes 50, that is, 5·10, while in the number 576, the same digit stands for five hundred, or 5·10·10. In Babylonian notation the number 60 was used as we use 10 in our number system, whence the name *sexagesimal* by which the Babylonian system is designated. Numbers less than 60 were denoted by two symbols: for unity and for ten. They were wedge-shaped (cuneiform) since the Babylonians wrote on clay tablets with a stylus having the form of a triangular prism. These signs were repeated as many times as needed, for example,

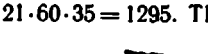




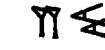
Numbers exceeding 60 were written in the following manner:  denoted $5 \cdot 60 + 2 = 302$, rather like our notation

of 52 denotes $5 \cdot 10 + 2$. The notation

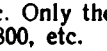
denoted the number $21 \cdot 60 \cdot 35 = 1295$. The following notation



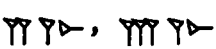
stood for $1 \cdot 60 \cdot 60 + 2 \cdot 60 + 5 = 3725$, much like the modern notation 125 denotes $1 \cdot 100 + 2 \cdot 10 + 5$. The sign was used as a placeholder, playing the part of zero. Thus, the notation

meant $2 \cdot 60 \cdot 60 + 0 \cdot 60 + 3 = 7203$. But the absence of any lower order digits was not indicated; for example, the number $180 = 3 \cdot 60$ was denoted as  , which is the same as the number 3. The same notation  might mean $10,800 = 3 \cdot 60 \cdot 60$, etc. Only the context could distinguish the numbers 3, 180, 10,800, etc.

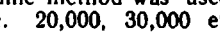
The notation  could also signify $\frac{3}{60}$, $\frac{3}{60 \cdot 60} = \frac{3}{3600}$, $\frac{3}{60 \cdot 60 \cdot 60} = \frac{3}{216,000}$ just as we use the numeral 3 to denote $\frac{3}{10}$, $\frac{3}{10 \cdot 10} = \frac{3}{100}$, $\frac{3}{10 \cdot 10 \cdot 10} = \frac{3}{1000}$, etc. in our system of decimal fractions. However, we readily differentiate between these fractions by annexing zeros in front of the 3, and we write $\frac{3}{10} = 0.3$, $\frac{3}{100} = 0.03$, $\frac{3}{1000} = 0.003$, etc. In the Babylonian notation these zero placeholders were not indicated.

Besides the sexagesimal system of numeration, the Babylonians used the decimal system, but it was not a positional system. Apart from symbols for 1 and 10, there were symbols for 100,  , 1000, , and 10,000, . The num-

bers 200, 300 and so on were written as



and so forth. The same method was used to write the numbers 2000, 3000 etc. 20,000, 30,000 etc. The number 274 looked like this:



the number 2068, like this:

etc.

The sexagesimal system originated at a later period than the decimal system because the numbers up to 60 were written on the basis of the decimal principle. It is still not known when and how the Babylonians developed the sexagesimal system. There are numerous hypotheses as to how this occurred but there is no firm proof for any of them.

The sexagesimal notation of whole numbers did not spread beyond the Assyrian-Babylonian empire, but sexagesimal fractions spread far and wide to the countries of the Near East, Central Asia, Northern Africa, and Western Europe. They found wide use, especially in astronomy, right up to the invention of decimal fractions (which was at the beginning of the 17th century). Traces of sexagesimal fractions are still found in the divisions of the degree of angle and arc (and also the hour) into 60 minutes, and of the minute into 60 seconds.

5. **Roman numeration.** The ancient Romans used a number system that is still in use and is called the Roman system of numeration. We use it for designating congresses and conferences, for numbering the introductory pages of books, chapter headings, etc.

In their latest form, the Roman numerals looked like this:

I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1000

The earlier forms were somewhat different. Thus, the number 1000 was denoted by the symbol (|), 500 by the symbol |).

There is no reliable information on the origin of the Roman numerals. The numeral V might have originally depict-

ed the human hand, the numeral X could have been built up out of two fives. In the same way, the symbol for 1000 could have developed out of doubling the sign for 500 (or vice versa).

The Roman system of numeration exhibits evident traces of the quinary system of numbers. But Latin (the language of the Romans) does not have a trace of the quinary number system. This must mean that these numerals were borrowed by the Romans from another people (most likely from the Etruscans).

All whole numbers (up to 5000) are written by means of iteration of the numerals listed above. If a large numeral precedes a smaller one, they are added, if the smaller one comes first (in which case the symbol is not repeated), then it is subtracted from the larger numeral (in Latin, the subtractive principle—see Sec. 18—is preserved in the names of two cardinal numbers: 18 and 19). For example, VI = 6, or 5 + 1, IV = 4, or 5 − 1, XL = 40, or 50 − 10, LX = 60, or 50 + 10. No digit is repeated more than three times: LXX = 70, LXXX = 80; the number 90 is written as XC (and not LXXXX).

The first 12 numbers are written in Roman numerals as follows:

I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII

**Examples:** XXVIII = 28, XXXIX = 39, CCCXCVII = 397, MDCCCXVIII = 1818

Performing arithmetical operations with multidigit numbers is an arduous task when done in Roman numerals. Nevertheless, Roman numerals were still the dominant number system in Italy up to the 13th century, and in other countries of Western Europe they persisted till the 16th century.

**6. The positional numeration of India.** The various regions of India had different number systems, one of which spread to other parts of the world and is today the generally accepted system of numeration. In this system, the numerals had the forms of the initial letters of the appropriate cardinal numbers in the ancient Indian language of Sanskrit (the Devanagari alphabet).

Originally, these symbols denoted the numbers 1, 2, 3, ..., 9, 10, 20, 30, ..., 90, 100, 1000, which in turn were used to write the other numbers. Later, a special sign (a heavy dot, circle) was introduced to indicate an empty position in a number. The signs for numbers exceeding 9 ceased to be used at a later period, and the Devanagari system of numeration

became the decimal positional system of numeration. It is not known how and when this conversion took place, but by the middle of the 8th century, the positional system of numeration was in wide use in India. It was about this time that it began to spread to other countries (Indochina, China, Tibet, into the territory of the present-day Central-Asian republics of the Soviet Union, Iran, and elsewhere). A decisive role in the spread of the Hindu numeration in the Arabic countries was played by a manual written at the beginning of the 9th century by Mohammed ibn-Musa al-Khowarizmi (from Khoresm—the present-day Khoresm Oblast of the Uzbek Republic of the USSR).* It was translated into Latin in Western Europe in the 12th century. In the 13th century, the Hindu system of numeration became dominant in Italy, and by the 16th century it spread to the other countries of Western Europe. The Europeans borrowed the Hindu number system from the Arabs and called it the Arabic system of numeration. Historically, this is not correct, but the name persists.

The Arabs also gave us the word "cipher" ("sifr" in Arabic) which literally means "empty position" (this is a translation of the Sanskrit "sunia" which has the same meaning).

The word was originally used to denote the empty position (as a placeholder) in a number and that meaning was still current in the 18th century, although in the 15th century the Latin term for "zero", "null" (nullum—nothing), had appeared.

The shapes of the Hindu numerals underwent a variety of modifications over the centuries; the form that we have today was established in the 16th century.

### 22. Names of Large Numbers

To facilitate reading and remembering large numbers, the digits are ordinarily grouped into periods of three each, which are separated by a comma or a space. Say, the number 35461298 is written: 35 461 298 (or 35,461,298). Here, the three digits 298 form the first period, 461, the second, and 35, the third.

* This remarkable scholar was also the founder of algebra (see Sec. 66). Mohammed wrote his works in Arabic, which in the East was the common language of learning, just as Latin was at one time in Western Europe. This explains the name al-Khowarizmi (which means a dweller of Khoresm) by which Mohammed is known in history.

Each digit has place value. From right to left we have: 8 is the units digit, 9 the tens digit, 2 the hundreds digit, 1 the thousands digit, 6 the ten thousands digit, etc.

The first period yields units, tens, hundreds, the second period, thousands, and the third period, millions.

The American system of numeration for denominations above one million is the same as the French system and the Russian system, and the British system corresponds to the German. In the American system each of the denominations above 1000 millions (the American billion) is 1000 times the one preceding (one trillion = 1000 billions; one quadrillion = 1000 trillions, one quintillion = 1000 quadrillions, one sextillion = 1000 quintillions, and so on for septillions, octillions, nonillions, etc.). In the British system the first denomination above 1000 millions (the British milliard) is 1000 times the preceding one, but each of the denominations above 1000 milliards (the British billion) is 1,000,000 times the preceding one (one trillion = 1,000,000 billions; one quadrillion = 1,000,000 trillions, etc.).

## 23. Arithmetic Operations

**1. Addition.** The concept of adding stems from such fundamentally elementary facts that it does not require a definition and cannot be defined in formal fashion. We can use synonymous expressions, if one so desires, and say it is the process of combining, and the like.

**Notation:** $8+3=11$; 8 and 3 are the *addends*, 11 is the *sum*.

**2. Subtraction.** When one number is subtracted from another the result is called the *difference* or *remainder*. The number subtracted is termed the *subtrahend*, and the number from which the subtrahend is subtracted is called the *minuend*.

**Notation:** $15-7=8$; 15 is the minuend, 7 is the subtrahend, and 8 is the remainder. Subtraction may be checked by addition: $8+7=15$.

**3. Multiplication.** Multiplication is the process of taking one number (called the *multiplicand*) a given number of times (this is the *multiplier*, which tells us how many times the multiplicand is to be taken). The result is called the *product*. The numbers multiplied together are called the factors of the product. (For multiplication by a fraction see Sec. 34).

**Notation:** $12 \times 5 = 60$ or $12 \cdot 5 = 60$; 12 is the multiplicand, 5 the multiplier, and 60 the product (here, 12 and 5 are the factors of the product). $12 \times 5 = 12 + 12 + 12 + 12 + 12$.

Nothing changes if we call 5 the multiplicand and 12 the multiplier. To take another example, $2 \times 5 = 2 + 2 + 2 + 2 + 2 = 10$ and $5 \times 2 = 5 + 5 = 10$. It is therefore more common to use the generic term "factor" for the multiplier and the multiplicand.

**4. Division.** Division is the process of finding one of two factors from the product and the other factor. It is the process of determining how many times one number is contained in another. The number divided by another is called the *dividend*. The number divided into the dividend is called the *divisor*, and the answer obtained by division is called the *quotient*.

**Notation:** $48 : 6 = 8$ (or $48 \div 6 = 8$). Here, 48 is the dividend, 6 the divisor, and 8 the quotient. Division may be checked by multiplication: the product of the divisor (6) and the quotient (8) yields the dividend. Division may also be written as $\frac{48}{6} = 8$ (see Sec. 36).

The quotient obtained by the division of one whole number by another one may not be a whole number, in which case the quotient may be indicated as fraction (Sec. 30). If the quotient is a whole number, we say that the first number is exactly divisible (or, simply, divisible) by the second number. For example, 35 is exactly divisible by 5 since the quotient is a whole number, 7.

In this case, the second number (5) is called the divisor of the first one, and the first number (35) is termed the multiple of the second one.

**Example 1.** 5 is a divisor of the numbers 25, 60, 80 and is not a divisor of 4, 13, 42, or 61.

**Example 2.** 60 is a multiple of 15, 20, 30 and is not a multiple of 17, 40, or 90.

In many cases it is possible to determine whether one number is divisible by another without actually performing the division (see Sec. 25).

In most cases there is a remainder after the division. In the process of division with a remainder we seek the largest whole number which, when multiplied by the divisor, yields a number that does not exceed the dividend. The desired number is a *partial quotient*. The difference between the dividend and the product of divisor by the partial quotient is termed the *remainder*, which is always less than the divisor.

**Example.** 19 is not divisible exactly by 5. The numbers 1, 2, 3 when multiplied by 5 yield 5, 10, 15, none of which exceeds 19, whereas 4 by 5 yields 20, which exceeds 19. Hence, the partial quotient is 3. The difference between 19 and the product $3 \cdot 5 = 15$ is $19 - 15 = 4$. The remainder is 4.

For division by zero, see Sec. 37.

**5. Involution** (raising a number to a power). To raise a number to an integral power (second, third, fourth, etc.) we multiply the quantity by itself two, three, four, etc. times (for raising a number to a negative, zero or fractional power, see Sec. 125). The number repeated as a factor is called the *base*; the number which indicates how many times the base is to be used as a factor is called the *exponent of the power*. The result is called the *power*.

**Notation:** $3^4 = 81$. Here, 3 is the base, 4 is the exponent, and 81 is the power; $3^4 = 3 \cdot 3 \cdot 3 \cdot 3$.

The second power is also called the *square*, the third power is also called the *cube* (of a number). The first power of a number is the number itself.

**6. Evolution** (finding the root of a number). Finding (or extracting) the root of a number is a process by which we find the base of a power from the power and the exponent. The power here is termed the *radicand*, the exponent is here the *index of the root*, and the desired base of the power is termed the *root*.

**Notation:** $\sqrt[4]{81} = 3$. Here, 81 is the radicand, 4 the index, and 3 the root. Raising 3 to the fourth power yields 81; $3^4 = 81$ (evolution can be checked by involution).

The second root is usually called the *square root*, the third root the *cube root*. When taking square roots the index 2 is usually omitted: $\sqrt{16} = 4$ means $\sqrt[2]{16} = 4$.

Addition and subtraction, multiplication and division, involution and evolution are pairs of *inverse operations*.

It is assumed that the reader is acquainted with the rules of the first four operations with whole numbers. Raising to a power is performed by means of repeated multiplication. For extraction of roots (evolution), see Secs. 58, 58a.

### 24. Order of Operations. Brackets

If several operations are performed one after another, the result, generally speaking, depends on the sequence (order) of the operations. For example, $4 - 2 + 1 = 3$ if the operations

are performed as indicated; but if we first add 2 and 1 and then subtract the sum from 4, we get 1. *Brackets* are used to indicate the sequence in which the operations are to be performed (in cases when the result depends on the order of the operations). Operations indicated in brackets are to be performed first. In our case, $(4-2)+1=3$, or $4-(2+1)=1$.

**Example 1.** $(2+4)\times 5-6\times 5=30$; $2+(4\times 5)=2+20=22$.

In order to avoid too many brackets, we agree not to indicate brackets: (1) when the operations of addition and subtraction are to be performed in the sequence in which they are indicated; for instance, in place of $(4-2)+1=3$ we write $4-2+1=3$; (2) when multiplication or division would be indicated in brackets; for instance, in place of $2+(4\times 5)=22$ we write $2+4\times 5=22$.

In computing expressions which either have no brackets or contain only one set of brackets, perform the operations in the following order: (1) first the operations in the brackets; multiplication and division in the sequence in which they are given but before addition and subtraction; (2) then the remaining operations, again multiplication and division being accomplished in the order in which they are indicated but prior to addition and subtraction.

**Example 2.** $2\cdot 5-3\cdot 3$. First multiply, $2\cdot 5=10$, $3\cdot 3=9$, then subtract $10-9=1$.

**Example 3.** $9+16:4-2\cdot(16-2\cdot 7+4)+6\cdot(2+5)$. First perform the operations indicated in the brackets:

$$16-2\cdot 7+4=16-14+4=6;\quad 2+5=7$$

Now handle the remaining operations:

$$9+16:4-2\cdot 6+6\cdot 7=9+4-12+42=43$$

It often happens that bracketed expressions themselves have to be enclosed in brackets, and the latter once more in brackets. In such cases, the *round brackets (parentheses)* are used first, then *square brackets*, [ ], and finally *curly brackets* (braces), { }. The sequence of operations then is as follows: perform all operations in the round brackets in the indicated order; then all computations in the square brackets by the same rules, and all the computations in the curly brackets, etc.; finally, the remaining operations are performed.

**Example 4.** $5+2\times[14-3\cdot(8-6)]+32:(10-2\cdot3)$. We carry out the operations in the round brackets: this yields

$$8-6=2; \quad 10-2\cdot3=10-6=4$$

the operations in the square brackets yield $14-3\cdot2=8$, and then the remaining operations

$$5+2\cdot8+32:4=5+16+8=29$$

which give us our answer.

**Example 5.** $\{100-[35-(30-20)]\}\cdot2$. The sequence of operations is: $30-20=10$; $35-10=25$; $100-25=75$; $75\cdot2=150$.

### 25. Criteria for Divisibility

**Divisibility by 2.** A number divisible by 2 is called an *even number*, otherwise it is *odd*. A number is divisible by two if its last digit is even or zero, otherwise it is not.

**Example.** The number 52,738 is divisible by 2 since the last digit is 8 (even); 7691 is not divisible by 2 because the last digit, 1, is odd; 1250 is divisible by 2 because the last digit is zero.

**Divisibility by 4.** A number is divisible by 4 if the last two digits are zeros or form a number divisible by 4, otherwise it is not so divisible.

**Examples.** 31,700 is divisible by 4 since the last two digits are zeros; 215,634 is not divisible by 4 because the last two digits form the number 34, which is not divisible by 4; 16,608 is divisible by 4 since the last two digits 08 yield the number 8 which is divisible by 4.

**Divisibility by 8.** A number is divisible by 8 if the last three digits are zeros or form a number divisible by 8. Otherwise it is not divisible by 8.

**Examples.** 125,000 is divisible by 8 (the last three digits are zeros); 170,004 is not divisible by 8 (the last three digits form the number 4, which cannot be divided by 8); 111,120 is divisible by 8 since the last three digits form 120, which is divisible by 8.

Similar criteria could be indicated for division by 16, 32, 64, etc., but they are of no practical value.

**Divisibility by 3 and by 9.** Only such numbers are divisible by 3 the sum of whose digits is divisible by 3; the

same for 9—the sum of the digits must be divisible by 9.

**Examples.** The number 17,835 is divisible by 3 but is not divisible by 9 since the sum of the digits $1+7+8+3+5=24$ is divisible by 3 but is not divisible by 9. The number 106,499 is not divisible either by 3 or by 9 since the sum of the digits (29) is not divisible by 3 or 9. The number 52,632 is divisible by 9 because the sum of the digits (18) is divisible by 9.

**Divisibility by 6.** A number is divisible by 6 if it is simultaneously divisible by 2 and by 3, otherwise it is not.

**Example.** 126 is divisible by 6 since it can be divided by 2 and by 3.

**Divisibility by 5.** Numbers ending in 0 or 5 are divisible by 5, otherwise they are not.

**Example.** 240 is divisible by 5 (the last digit is zero); 554 is not divisible by 5 (the last digit is 4).

**Divisibility by 25.** A number is divisible by 25 if the last two digits are zeros or form a number that is divisible by 25 (such are 00, 25, 50, or 75), otherwise it is not divisible by 25.

**Example.** 7150 is divisible by 25 (it ends in 50); 4855 is not divisible by 25.

**Divisibility by 10, by 100 and by 1000.** Only numbers ending in 0 are divisible by 10, only those ending in two zeros are divisible by 100, and only those ending in three zeros are divisible by 1000.

**Examples.** 8200 is divisible by 10 and by 100; 542,000 is divisible by 10, 100 and 1000.

**Divisibility by 11.** The number 11 divides only those numbers whose sum of digits occupying odd positions is either equal to the sum of the digits occupying even positions or differs from it by a number which is divisible by 11.

**Examples.** The number 103,785 is divisible by 11 since the sum of the digits in odd positions ($1+3+8=12$) is equal to the sum of the digits in even positions ($0+7+5=12$). The number 9,163,627 is divisible by 11 since the sum of digits in odd positions is $9+6+6+7=28$, while the sum of digits in even positions is $1+3+2=6$; the difference between 28 and 6 is 22, which is divisible by 11. The number 461,025 is not divisible by 11 since the numbers $4+1+2=7$ and $6+0+5=11$ are not equal and their difference $11-7=4$ is not divisible by 11.

There exist divisibility criteria for other numbers besides those given above but they are more complicated.

### 26. Prime and Composite Numbers

All whole numbers (integers) other than 1 have at least two divisors: unity (one) and the number itself. Numbers that do not have any other divisors are called *prime numbers*. For example, 7, 41, 53 are prime numbers. Numbers which have other divisors besides 1 and the number itself are called *composite numbers*. An example is 21 with its divisors 1, 3, 7, 21 and 81 with its divisors 1, 3, 9, 27, 81. The number one (unity) could be classed as a prime, but it is better to put it in a separate class outside the prime and composite numbers (this convention stems from the fact that many of the rules which hold true for all other primes are invalid when applied to unity).

There exist infinitely many prime numbers.

The primes not exceeding 200 are (see pages 50-51 for the Table of Primes $< 6000$):

$$\left.\begin{array}{l} 2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43, \\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97,\ 101, \\ 103,\ 107,\ 109,\ 113,\ 127,\ 131,\ 137,\ 139,\ 149,\ 151, \\ 157,\ 163,\ 167,\ 173,\ 179,\ 181,\ 191,\ 193,\ 197,\ 199 \end{array}\right\} \quad (A)$$

### 27. Factorization into Prime Factors

Every composite number can be represented uniquely as a product of prime factors. For example, $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$; $45 = 3 \cdot 3 \cdot 5 = 3^2 \cdot 5$ (or $3^2 \cdot 5^1$), $150 = 2 \cdot 3 \cdot 5 \cdot 5 = 2 \cdot 3 \cdot 5^2$ (or $2^1 \cdot 3^1 \cdot 5^2$). For small numbers, factorization can be accomplished by mere guesswork. For large numbers the following technique can be used.

**Example 1.** Suppose we have a number 1421. Taking the primes, one after the other, out of Table of Primes (A), we stop at the prime which is a divisor of the given number. Using the divisibility criteria we see that the numbers 2, 3, 5 cannot be divisors of 1421; we attempt to divide by 7 and we see that 1421 is divisible by 7 yielding a quotient of 203. To the left of the line we write 1421; on the right the divisor; under the number we write the quotient 203. Then we

| | | |
|---|---|---|
| **Work:** 1421 | 7 | test 203 in the same manner. We ignore the |
| 203 | 7 | numbers 2, 3, 5 that proved ineffective in |
| 29 | 29 | the first trial and begin |

with 7. It turns out that 7 is a divisor of 203. Write it down to the right of 203. Under 203 write the quotient 29, which

is prime. This completes the factorization. We have:

$$1421 = 7 \cdot 7 \cdot 29 = 7^2 \cdot 29$$

The general technique may be simplified in a number of cases.

**Example 2.** Let us factor the number 1,237,600 into prime factors. Noting that $1{,}237{,}600 = 12{,}376 \times 100$, we factor these two factors separately. The latter is readily decomposable into $100 = 10 \cdot 10 = 2 \cdot 5 \cdot 2 \cdot 5 = 2^2 \cdot 5^2$. Then we factor the first one as follows.

| | | | |
|---|---|---|---|
| **Work:** | 12,376 | 2 | From Table (A) we take the first prime, 2; we immediately see that it is a divisor of 12,376. Finding the quotient 6188, we again take 2 from Table (A). The second quotient 3094 is also even, so we divide it by 2. The result 1547 |
| | 6 188 | 2 | |
| | 3 094 | 2 | |
| | 1 547 | 7 | |
| | 221 | 13 | |
| | 17 | 17 | |

is no longer divisible by 2. The divisibility criteria show that it is not divisible by 3 or by 5 either. Let us try 7. We get the quotient 221. Try 7 again. It is not divisible. Then try the following primes. 221 is not divisible by 11 but it is divisible by 13 with the quotient 17, which is prime.

The final result is: $1{,}237{,}600 = 2^3 \cdot 7 \cdot 13 \cdot 17 \cdot 2^2 \cdot 5^2 = 2^5 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$.

### 28. Greatest Common Divisor

A *common divisor* of several numbers is a divisor (see Sec. 23, Item 4) of each of them. For example, the numbers 12, 18, 30 have 3 as a common divisor; 2 is also a common divisor. In every set of common divisors there is always a *greatest common divisor* (6 in our example), or G.C.D.

**Examples.** 4 is the G.C.D. of 16, 20, 28; 5 is the G. C. D. of 5, 30, 60, and 90.

For small numbers, the G.C.D. can be found by inspection. If we have to do with large numbers, decompose each into prime factors (see Sec. 27) and write out those which are factors of all the given numbers. Take each factor with the smallest exponent with which it is contained in the given numbers. Then multiply.

**Example 1.** Find the G.C.D. of 252, 441, and 1080. Factor into prime factors.

$$252 = 2^2 \cdot 3^2 \cdot 7, \quad 441 = 3^2 \cdot 7^2, \quad 1080 = 2^3 \cdot 3^3 \cdot 5$$

We have only one common factor, the prime factor 3. The smallest exponent of this factor is 2. The G.C.D. is $3^2=9$.

**Example 2.** Find the G.C.D. of the numbers 234, 1080, and 8100. $234=2\cdot3^2\cdot13$, $1080=2^3\cdot3^3\cdot5$, $8100=2^2\cdot3^4\cdot5^2$

$$\text{G. C. D.}=2\cdot3^2=18$$

It may happen that there are no prime factors common to all the numbers. Then the greatest common divisor is 1. For instance, for $15=3\cdot5$, $10=2\cdot5$, $6=2\cdot3$ the G.C.D. is 1. If the G.C.D. of two numbers is 1, then these numbers are called relatively prime. To illustrate, 15 and 22 are relatively prime numbers.

### 29. Least Common Multiple

A *common multiple* of several numbers is a multiple of each of them (Sec. 23, Item 4). The numbers 15, 6, and 10 have 180 as a common multiple. The number 90 is also a common multiple of these numbers. The set of common multiples has a *least common multiple* (30 in our case), or L.C.M. When dealing with small numbers, the L.C.M. can be seen at once. If the number is large, do as follows: factor the given numbers into prime factors; write out all prime factors of at least one of the given numbers; then take each factor and raise it to the highest power that it is contained in the given numbers. Then multiply.

**Example 1.** Find the L.C.M. of 252, 441, 1080.

Factor into prime factors: $252=2^2\cdot3^2\cdot7$, $441=3^2\cdot7^2$, $1080=2^3\cdot3^3\cdot5$. Multiply out $2^3\cdot3^3\cdot7^2\cdot5$ and we find the L.C.M. to be 52,920.

**Example 2.** Find the L.C.M. of 234, 1080, 8100 (see Sec. 28, Example 2). The L.C.M. $=2^3\cdot3^4\cdot5^2\cdot13=210{,}600$.

### 30. Common Fractions

A *common fraction* (or, simply, a fraction) is a part of unity or several equal parts of unity. The number which indicates how many parts a unit is divided into is called the *denominator* of the fraction; the number indicating how many parts are taken is the *numerator* of the fraction.

**Notation:** $\frac{3}{5}$ or 3/5 (three fifths): here, 3 is the numerator and 5 is the denominator.

If the numerator is less than the denominator, the fraction is less than unity and is called *a proper fraction:* $\frac{3}{5}$ is a proper fraction. If the numerator is equal to the denominator, the fraction is equal to unity. If the numerator is greater than the denominator, the fraction exceeds unity. In these latter two cases it is called an *improper fraction*. For example, $\frac{5}{5}$, $\frac{17}{5}$ are improper fractions. In order to take out the largest whole number contained in an improper fraction, divide the numerator by the denominator. If the division is exact, then the improper fraction is equal to the quotient. Say, $\frac{45}{5}=45:5=9$. If the division is not exact, the (partial) quotient yields the desired integer, while the remainder becomes the numerator of the fractional part, and the denominator of the fractional part remains unchanged.

**Example.** Given the fraction $\frac{48}{5}$. Divide 48 by 5 to get the quotient 9 and a remainder of 3; $\frac{48}{5}=9\frac{3}{5}$.

A number consisting of an integral part (whole number) and a fractional part (fraction) is called a *mixed number* $\left(9\frac{3}{5} \text{ for instance}\right)$. The fractional part of a mixed number may be an improper fraction $\left(\text{like } 7\frac{13}{5}\right)$. It is then possible to take out the largest whole number (see above) and represent the mixed number so that the fraction becomes proper (or disappears altogether). For example,

$$7\frac{13}{5}=7+\frac{13}{5}=7+2\frac{3}{5}=9\frac{3}{5}$$

Mixed numbers are usually reduced to such form.

If is often necessary (in multiplying fractions, say) to operate in reverse: given a mixed number, it is required to represent it in the form of a fraction (improper fraction). To do this, (1) multiply the integer in the mixed number by the denominator of the fractional part; (2) add the numerator to the product. The resulting number will be the numerator of the desired fraction, and the denominator remains unchanged.

**Example.** Given the mixed number $9\frac{3}{5}$. (1) $9 \cdot 5 = 45$; (2) $45 + 3 = 48$; (3) $9\frac{3}{5} = \frac{48}{5}$

### 31. Reducing Fractions

The value of a fraction is not changed if the numerator and denominator are multiplied by the same number. For example

$$\frac{3}{5} = \frac{3 \cdot 6}{5 \cdot 6} = \frac{18}{30}, \quad \frac{1}{2} = \frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6}, \quad \frac{1}{2} = \frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$$

This is called *reducing (changing) the fraction to higher terms.*

The value of a fraction remains unchanged if the numerator and denominator are divided by one and the same number. For example

$$\frac{18}{30} = \frac{18:6}{30:6} = \frac{3}{5}, \quad \frac{4}{8} = \frac{4:4}{8:4} = \frac{1}{2}$$

This is called *reducing the fraction to lower terms*, or simply reducing the fraction. We say that $\frac{3}{5}$ is obtained from $\frac{18}{30}$ by dividing out 6.

A fraction can only be reduced if the numerator and denominator have the same divisors (that is, if they are not relatively prime). The reduction may be accomplished gradually or at once using the G.C.D.

**Example.** Reduce the fraction $\frac{108}{144}$ to lower terms. Using the divisibility-by-4 criterion (see Sec. 25 above), we see that 4 is a common divisor of the numerator and denominator. Dividing out 4 we have $\frac{108}{144} = \frac{108:4}{144:4} = \frac{27}{36}$. Noting that 27 and 36 have 9 as a common divisor, we divide 9 out of $\frac{27}{36}$: $\frac{27}{36} = \frac{3}{4}$. Further reduction is impossible since 3 and 4 are relatively prime.

We get the same result if we find the greatest common divisor of the numbers 108 and 144. It is 36. Dividing out 36 we obtain

$$\frac{108}{144} = \frac{108:36}{144:36} = \frac{3}{4}$$

Dividing by the greatest common divisor we have the *fraction in lowest terms.*

## 32. Comparing Fractions.

### Finding a Common Denominator

If two fractions have the same numerator, the larger fraction is that with the smaller denominator. For example, $\frac{1}{3} > \frac{1}{4}$, $\frac{5}{7} > \frac{5}{9}$. If two fractions have the same denominator, the greater fraction is that with the greater numerator: $\frac{5}{8} > \frac{3}{8}$.

In order to compare two fractions with different numerators and denominators, either one or both of the fractions have to be transformed so that the denominators are the same. To do this, change the first fraction to the terms of the denominator of the second, and the second to the denominator of the first.

**Example.** Compare the fractions $\frac{3}{8}$ and $\frac{7}{12}$. Change the first by a factor of 12, the second by 8. This yields $\frac{3}{8} = \frac{36}{96}$, $\frac{7}{12} = \frac{56}{96}$. Now the denominators are the same. Comparing the numerators we see that the second fraction is greater than the first.

This process of changing fractions is called *finding a common denominator*.

To reduce several fractions to a common denominator, we can change the denominator of each by a factor equal to the product of the other denominators. For instance, in order to reduce the fractions $\frac{3}{8}$, $\frac{5}{6}$, $\frac{2}{5}$ to a common denominator, change the first denominator by $5 \cdot 6 = 30$, the second by $8 \cdot 5 = 40$, the third by $8 \cdot 6 = 48$. We get $\frac{3}{8} = \frac{90}{240}$, $\frac{5}{6} = \frac{200}{240}$, $\frac{2}{5} = \frac{96}{240}$. The common denominator is the product of the denominators of all the given fractions: $8 \cdot 6 \cdot 5 = 240$.

This method of finding a common denominator is the simplest and, in many cases, the most practical. The sole inconvenience is that the common denominator may turn out to be too big, whereas smaller ones may be available. Namely, for a common denominator we can take any common multiple (the least common multiple, say) of the given denominators. It is then necessary to change (multiply) each fraction by the quotient obtained from dividing the common

multiple by the denominator of the given fraction (this quotient will be called the *additional factor*).

**Example.** Given the fractions $\frac{3}{8}$, $\frac{5}{6}$, $\frac{2}{5}$. The least common multiple of the denominators 8, 6, 5 is 120. The additional factors are: $120:8=15$, $120:6=20$, $120:5=24$. Change (multiply) the first fraction by 15, the second by 20, the third by 24:

$$\frac{3}{8}=\frac{45}{120},\quad \frac{5}{6}=\frac{100}{120},\quad \frac{2}{5}=\frac{48}{120}$$

Most arithmetics give only this technique for reducing to a common denominator. Actually, it is of practical use only when the L.C.M. is seen by inspection, otherwise a great deal of time is needed to find the L.C.M. and the additional factors. What is more, it often happens that the L.C.M. is not much less than the product of the denominators or not less at all. But then a lot of time is spent for no good reason.

### 33. Adding and Subtracting Fractions

If the denominators of the fractions are the same, the fractions may be added by adding their numerators; to subtract them, subtract the numerator of the subtrahend from the numerator of the minuend. This sum or difference will be the numerator of the answer, the denominator remaining unchanged. If the denominators differ, first reduce the fractions to a common denominator.

**Example 1.** $\frac{5}{8}+\frac{7}{8}=\frac{12}{8}=1\frac{4}{8}=1\frac{1}{2}$.

**Example 2.** $\frac{3}{8}+\frac{5}{6}-\frac{2}{5}=\frac{45}{120}+\frac{100}{120}-\frac{48}{120}=\frac{97}{120}$.

To add mixed numbers, separately find the sum of the integral parts and the fractional parts.

**Example 3.** $7\frac{3}{4}+4\frac{5}{6}=(7+4)+\left(\frac{3}{4}+\frac{5}{6}\right)=11\frac{19}{12}$

$=12\frac{7}{12}$.

When subtracting mixed numbers, the fractional part of he subtrahend may be larger than the fractional part of the

minuend. In that case, borrow unity from the minuend and change to an improper fraction.

**Example 4.** $7\frac{1}{4}-4\frac{1}{3}=7\frac{3}{12}-4\frac{4}{12}=6\frac{15}{12}-4\frac{4}{12}=2\frac{11}{12}$.

**Example 5.** $11-10\frac{5}{7}=10\frac{7}{7}-10\frac{5}{7}=\frac{2}{7}$.

### 34. Multiplication of Fractions. Definition

For multiplication and division of a fraction by a whole number (integer), the definitions given above in Sec. 23 (Items 3 and 4) hold true. For example,

$$2\frac{3}{4}\times 3=2\frac{3}{4}+2\frac{3}{4}+2\frac{3}{4}=8\frac{1}{4}$$

Conversely, $8\frac{1}{4}:3=2\frac{3}{4}$. The practical rules of computation are given below.

The definition of Sec. 23 is not valid for multiplication by a fraction. For example, the operation $2\frac{1}{2}\cdot\frac{3}{4}$ cannot be carried out if it is understood that $2\frac{1}{2}$ is to be taken as a factor $\frac{3}{4}$ times.

To multiply a given number (integer or fraction) by a fraction means to divide the number by the denominator of the fraction and multiply the result by the numerator.

**Example.** $800\cdot\frac{3}{4}$, $800:4=200$ $200\cdot 3=600$ so that $800\cdot\frac{3}{4}=600$. The sequence of operations (division and multiplication) may be reversed and the result will be the same: $800\cdot 3=2400$ and $2400:4=600$.

The above definition is not a mere whim, it follows from the necessity to preserve intact the role that multiplication plays in practice and theory when we deal with whole numbers. Two examples will suffice to make this clear.

**Example.** A litre of kerosene weighs 800 grams. What is the weight of 4 litres? **Solution:** $800\cdot 4=3200$(g) $=3$ kg 200 g. The result is found by multiplying by 4.

What does $\frac{3}{4}$ of a litre of kerosene weigh? **Solution:** $800\cdot\frac{3}{4}=600$ (g) (see preceding example).

If we define multiplication by a fraction differently, the answer will not be correct. If we proceeded from the definition given in Sec. 23 and regarded multiplication by $\frac{3}{4}$ as impossible, then we would have to solve the problem of the weight of the kerosene by different operations: multiplication for an integral number of litres, and via a different operation for a fractional number of litres.*

In the multiplication of whole numbers, the product remains the same no matter what the order of the factors: $3 \cdot 4 = 4 \cdot 3 = 12$. This property is preserved in multiplication by a fraction as well. For example, $\frac{2}{3} \cdot 3 = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2$. This result is obtained on the basis of the earlier definition (see Sec. 23). Interchange the factors: $3 \cdot \frac{2}{3}$. The latter definition of multiplication no longer holds true, but the new definition yields $3 \cdot \frac{2}{3} = 2$.

Generally speaking, it turns out that under the new definition of multiplication, all earlier properties and rules remain valid, with the exception of one: in the earlier definition of multiplication, a number increased in value, whence the name "multiplication" ("multus" = many). Now we have to say that in multiplication by a number exceeding unity the multiplicand increases, and in multiplication by a number less than unity (a proper fraction), the multiplicand decreases. This discrepancy between the latter fact and the name of the operation is due to the fact that the term "multiplication" originated at so remote a time that the concept of multiplication referred solely to whole numbers.

### 35. Multiplication of Fractions. Rule

*To multiply a fraction by a fraction, multiply the numerators together for the numerator of the product and multiply the denominators together for the denominator of the product.* If there are mixed numbers, convert them to improper fractions before multiplying. Also, before multiplying, divide

* The question naturally arises as to whether it is possible to give a definition suitable for multiplication by an integer and by a fraction. This appears to be impossible: when defining multiplication by a fraction we unavoidably have to assume as known multiplication by a whole number (see definition of this section).

out (cancel) any common factors in the numerator and the denominator.

**Example 1.** $2\frac{1}{12}\cdot 1\frac{7}{20}=\frac{25}{12}\cdot\frac{27}{20}=\frac{5}{4}\cdot\frac{9}{4}=\frac{45}{16}=2\frac{13}{16}$ (5 is divided out of 25 and 20, 3 out of 12 and 27).

The foregoing is extended to the case when the number of factors exceeds two.

**Example 2.** $4\frac{1}{2}\cdot\frac{4}{7}\cdot 4\frac{2}{3}=\frac{9\cdot 4\cdot 14}{2\cdot 7\cdot 3}=\frac{3\cdot 2\cdot 2}{1\cdot 1\cdot 1}=12$
(3 is divided out of 9 and 3, 2 out of 4 and 2, 7 out of 14 and 7).

If some of the factors are whole numbers, then each one is regarded as a fraction with denominator 1.

**Example 3.** $\frac{5}{8}\cdot 7\cdot\frac{4}{15}=\frac{5\cdot 7\cdot 4}{8\cdot 1\cdot 15}=\frac{1\cdot 7\cdot 1}{2\cdot 1\cdot 3}=\frac{7}{6}=1\frac{1}{6}$
(5 is divided out of 5 and 15, 4 out of 4 and 8).

### 36 Division of Fractions

The definition of division given earlier in Sec. 23 holds true for the division of fractions as well. From it follows the rule:

*To divide a number by a fraction, multiply the number by the reciprocal of the fraction* (the reciprocal is the fraction formed by interchanging the numerator and denominator: the reciprocal of $\frac{6}{7}$ is $\frac{7}{6}$).

**Example 1.** $\frac{2}{3}:\frac{4}{15}$. The reciprocal of $\frac{4}{15}$ is $\frac{15}{4}$. Hence $\frac{2}{3}:\frac{4}{15}=\frac{2\cdot 15}{3\cdot 4}=2\frac{1}{2}$.

**Example 2.** $1\frac{3}{5}:3\frac{1}{5}=\frac{8}{5}:\frac{16}{5}=\frac{8\cdot 5}{5\cdot 16}=\frac{1\cdot 1}{1\cdot 2}=\frac{1}{2}$.

This rule is also applicable when the dividend and divisor are whole numbers. For example, $2:5=2\cdot\frac{1}{5}=\frac{2}{5}$. Thus, the fraction bar is equivalent to the division sign.

### 37. Operations Involving Zero

**Addition.** Adding zero to any number leaves the number unchanged: $5+0=5$; $3\frac{5}{7}+0=3\frac{5}{7}$.

**Subtraction.** Subtracting zero from any number leaves the number unchanged: $5-0=5$; $3\frac{5}{7}-0=3\frac{5}{7}$.

**Multiplication.** Zero times any number is always zero: $5 \cdot 0 = 0$; $0 \cdot 3\frac{5}{7} = 0$; $0 \cdot 0 = 0$.

**Division.** 1. The quotient obtained by the division of zero by any nonzero number is zero: $0 : 7 = 0$; $0 : \frac{3}{5} = 0$.

2. The quotient resulting from the division of zero by zero is indeterminate. Here, any number satisfies the definition of a quotient (see Sec. 23, Item 4). For example, we could set $0 : 0 = 5$ because $5 \cdot 0 = 0$, but we could also put $0 : 0 = 3\frac{5}{7}$ because $3\frac{5}{7} \cdot 0 = 0$. Thus, the problem of dividing zero by zero has infinitely many solutions and is meaningless without further information, which has to indicate how the dividend and divisor varied before they became zero. If this is known, then in most cases it is possible to give meaning to the expression 0:0. For instance, if we know that the dividend took on the successive values $\frac{3}{100}, \frac{3}{1000}, \frac{3}{10,000}$, etc., and the divisor assumed the values $\frac{7}{100}, \frac{7}{1000}$, etc., then the quotient, meantime, assumed the values $\frac{3}{100} : \frac{7}{100} = \frac{3}{7}; \frac{3}{1000} : \frac{7}{1000} = \frac{3}{7}$, etc., which is to say, remained equal to $\frac{3}{7}$, and so the quotient of 0:0 can, here, be taken equal to $\frac{3}{7}$.

In such cases, one speaks of evaluating the indeterminate expression 0:0 (see Sec. 217, Example 2). Higher mathematics offers a number of techniques for evaluating the indeterminate expression 0:0, but in certain cases the tools of elementary mathematics suffice.

3. The quotient obtained by dividing some nonzero number by zero does not exist because in this case no number can satisfy the definition of a quotient (see Sec. 23, Item 4).

As an example, take 7:0. No matter what number we take to test this out (say, 2, 3, 7) we get the same unsatisfactory answer ($2 \cdot 0 = 0$, $3 \cdot 0 = 0$, $7 \cdot 0 = 0$) whereas what we need is 7. We can say that the problem of dividing a nonzero number by zero has no solution.

On the other hand, a nonzero number may be divided by a number arbitrarily close to zero, and the closer the divisor is to zero, the greater will be the quotient. Thus, if we divide $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10,000}$, etc., by 7, we obtain the quo-

tients 70, 700, 7000, 70,000, etc., which increase without bound. For this reason, we say that the quotient obtained by dividing 7 by 0 is "infinitely great" or is "equal to infinity", and we write: $7:0=\infty$. The meaning of this expression is that if the divisor approaches zero and the dividend remains equal to 7 (or approaches 7), then the quotient will increase without bound.

### 38. The Whole and a Part

1. **Finding a part from the whole.** To find some part of a number, multiply it by the fraction expressing that part.

**Example.** A committee of 120 members meets for elections. Two thirds of the body must be present. How many members must be present for the meeting to take place?

**Solution:** $120\cdot\frac{2}{3}=80$.

2. **Finding the whole from a part.** To find a number when one of its parts is known, divide the number by the fraction expressing the given part.

**Example.** The dead weight of an ox is 3/5 the live weight. What is the live weight of an ox whose dead weight is found to be 420 kg?

**Solution:** $420:\frac{3}{5}=700$ (kg).

3. **Expressing a part as a fraction of the whole.** To express a part as a fraction of the whole, divide by the whole number.

**Example.** Four students are absent in a class of 30. What part of the class is absent?

**Solution:** $4:30=\frac{4}{30}=\frac{2}{15}$.

### 39. Decimal Fractions

Computations involving common fractions become very unwieldy if the denominators are big numbers. The main difficulty lies in reducing the fraction to a common denominator. This is because there is no system in the choice of denominator, any number will do. That is why even in antiquity the idea arose of choosing regularly (not arbitrarily) certain parts of a unit (in common fractions they play the

part of denominators). The most ancient systematic fractions which were used in Babylonia some 4000 years B.C. and were passed on by the ancient Greek astronomers to the astronomers of Western Europe were the sexagesimal fractions (see Sec. 21, Item 4). At the end of the 16th century when intricate computations involving fractions were widely used in all spheres of life, systematic fractions of a different kind came into use: decimal fractions (see Sec. 45). Here, the unit is divided into ten equal parts (tenths), each tenth into tenths (hundredths), and so on. The advantage of decimal fractions over other systematic fractions lies in the fact that they are based on the same number scale (of ten) as are the whole numbers. As a result, both notation and rules of handling decimals are essentially the same as those used when dealing with whole numbers (integers).

When writing decimals there is no need to indicate the name of the parts (denominator); it is clear from the position of the appropriate digit. We first write the integral part, then a dot (decimal point),* then the first digit after the decimal point represents tenths, the second digit, hundredths, the third, thousandths, etc. The digits which come after the decimal point on the right are termed decimals (decimal places).

**Example.** 7.305 signifies seven units, 3 tenths, 5 thousandths (the zero indicates that there are no hundredths), or $7.305 = 7 + \frac{3}{10} + \frac{0}{100} + \frac{5}{1000}$.

One of the advantages of decimal fractions is that the expression of the fractional part is given directly in a form reduced to a common denominator:

$$7.305 = 7\frac{305}{1000}$$

the number following the decimal point (305) is the numerator of the fractional part and the denominator of the fraction is the number which shows how many parts are indicated by the last decimal place (in our case it is 1000).

If a decimal fraction does not have an integral part, then the best usage recommends a zero to be placed before the decimal point; for example: $\frac{35}{100} = 0.35$.

* In some countries, a comma is used to separate the whole number from the decimal fraction.

### 40. Properties of Decimal Fractions

1. Annexing zeros to the right of a decimal fraction does not change the value.

**Example.** $12.7 = 12.70 = 12.700$, etc.

(for the difference that is made between 12.7 and 12.70 see Sec. 48).

2. Dropping the zeros at the end of a decimal fraction does not change the value of the fraction.

**Example.** $0.00830 = 0.0083$. (Zeros that do not come at the end of the fraction cannot be omitted).

3. A decimal fraction is increased (multiplied by) 10, 100, 1000, etc. times if the decimal point is moved one place, two places, three places, etc. to the right.

**Example.** The number 13.058 becomes 100 times larger if we write 1305.8.

4. A decimal fraction is reduced dy 10, 100, 1000, etc. times if the decimal point is moved leftwards one, two, three, etc. places.

**Example.** 176.24 decreases 10 times if we write 17.624; it will be 1000 times less if we write 0.17624.

These properties enable us to perform rapid multiplication and division by the numbers 10, 100, 1000, etc.

**Examples.** $12.08 \cdot 100 = 1208$, $12.08 \cdot 10{,}000 = 120{,}800$ (first write 12.08 as 12.0800 and then move the decimal point to the right four places); $42.03 : 10 = 4.203$, $42.03 : 1000 = 0.04203$ (first write 42.03 as 0042.03 and then move the decimal point three places to the left).

### 41. Addition, Subtraction and Multiplication of Decimal Fractions

Addition and subtraction of decimal fractions are performed in the same way as the addition and subtraction of whole numbers; take care to write each digit in its proper place (tenths under tenths, hundredths under hundredths, and the like).

**Work:**

$$\begin{array}{r} 2.3 \\ +\ 0.02 \\ 14.96 \\ \hline 17.28 \end{array}$$

**Example.** $2.3 + 0.02 + 14.96 = 17.28$.

**Multiplication of decimal fractions.** Multiply the given numbers as whole numbers disregarding the decimal point. Then insert the decimal point using the following rule: *take the sum of the decimal places in all factors and point off that number of decimal places in the product.*

**Example 1.** $2.064 \cdot 0.05$. Multiply the whole numbers $2064 \cdot 5 = 10{,}320$. The first factor had three decimal places, the second two, and so the product must have five places set off after the decimal point. This yields 0.10320. The zero at the end of the fraction is dropped and we have $2.064 \cdot 0.05 = 0.1032$.

In this method, do not drop zeros before pointing off the decimal places (when multiplying by the method given in Sec. 55, the zeros may be dropped).

**Example 2.** $1.125 \cdot 0.08$; $1125 \cdot 8 = 9000$. The number of decimal places is $3 + 2 = 5$. Annexing zeros to the left of 9000 (009000) we point off five decimal places to get $0.09000 = 0.09$.

### 42. Division of a Decimal Fraction by an Integer

1. If the dividend is less than the divisor, write a zero in the integral part of the quotient and then the decimal point. Then, disregarding the decimal point, annex to the integral part of the dividend the first digit of its fractional part; if the resulting number is less than the divisor, put a zero after the decimal point and annex another digit of the dividend; if we still have a number less than the divisor, put down another zero, etc., until we get a number exceeding the divisor. The division is then performed in the same way as for whole numbers (integers). Note that the dividend may be "expanded" without bound rightwards from the decimal point by adding zeros.

**Note.** The process of division, as described above, may never end. In that case, the quotient cannot be expressed exactly by means of a decimal fraction. But we can terminate the process at any point and obtain an approximate result (see Sec. 44 below).

**Example 1.** 13.28:64.

**Work:**

| | |
|---|---|
| 13.28 | 64 |
| 12.8 | 0.2075 |
| 48 | |
| 480 | |
| 448 | |
| 320 | |

Here the number 132 exceeds the divisor as soon as we shift the decimal point one place to the right, and so there is no zero directly after the decimal point. But after we bring down the next digit the first remainder (48) is less than the divisor, so we put a zero in the dividend (expanded, the dividend becomes 13.280). This zero is then brought down and we can continue the division process. We get a remainder of 32 and again have to bring down a zero (the dividend then becomes 13.2800).

**Example 2.** 0.48:75.

**Work:**

| | |
|---|---|
| 0.480 | 75 |
| 450 | 0.0064 |
| 300 | |

Here, moving the decimal point one place to the right we get 4, which is less than 75; write zero in the quotient after the decimal point; move the decimal point one more place to the right in the dividend to get 48, which is still less than 75. Put a second zero in the quotient to the right of the decimal point. Adding one zero to the fraction we get 0.480, etc.

2. If the dividend is larger than the divisor, first divide the integral part; write down the result in the quotient and place the decimal point. The division process then proceeds as in the preceding case.

**Example 3.** 542.8:16.

**Work:**

| | |
|---|---|
| 542.8 | 16 |
| 48 | 33.925 |
| 62 | |
| 48 | |
| 148 | |
| 144 | |
| 40 | |
| 32 | |
| 80 | |

Dividing the integral part, we get a quotient of 33 and a remainder (the second remainder) of 14. Put the decimal point after 33 and bring down the next digit, 8. Divide 16 into 148 to get 9, which is the first digit after the decimal point, etc.

The same procedure is used in dividing a whole number by a whole number if it is desired to give the quotient as a decimal fraction.

**Example 4.** 417:15.

**Work:**

| | |
|---|---|
| 417 | 15 |
| 30 | 27.8 |
| 117 | |
| 105 | |
| 120 | |

Here, the decimal point is inserted after the last integral remainder (12) is obtained. The dividend 417 may be written as 417.0; it is then represented as a decimal.

***Alternative method of division.***

The process of division can be written differently:

**Example 5:** Divide 45, 837 by 312

```
      14691
  312| 45837
     312
     ----
     1463
     1248
     ----
      2157
      1872
      ----
      2850
      2808
      ----
       420
       312
       ---
       108 (remainder)
```

### 43. Division Involving Decimal Fractions

To divide a decimal fraction (or a whole number) by a decimal fraction, drop the decimal point in the divisor and move the decimal point rightwards in the dividend the same number of places as in the fractional part of the divisor (if necessary, annex zeros at the end of the dividend). Division can now be performed as indicated in the preceding section.

**Example.** 0.04569:0.0012.

**Work:**

```
456.9  | 12
36     | 38.075
-----
 96
 96
 --
 90
 84
 --
 60
```

There are four decimal places in the fractional part of the divisor, and so we move the decimal point four places to the right in the dividend to get 456.9. Now **divide 456.9 by 12.**

### 44. Changing a Decimal Fraction to a Common Fraction and Vice Versa

To change a decimal fraction to a common fraction, drop the decimal point and make the resulting number the numerator of the fraction; the denominator is the number indicated by the last decimal place. It is desirable, if possible, to reduce the fraction to lowest terms.

If the decimal exceeds unity, it is best to change only the decimal part to a common fraction and leave the integral part unchanged.

**Example 1.** Change 0.0125 to a common fraction. The last decimal, 5, is in the ten-thousandths place, and so the denominator will be 10,000 We have $0.0125 = \frac{125}{10,000} = \frac{1}{80}$.

**Example 2.** $2.75 = 2\frac{75}{100} = 2\frac{3}{4}$, or $2.75 = \frac{275}{100} = \frac{11}{4}$. The former procedure is to be preferred; that is, leave unchanged the 2 to the left of the decimal point and change 0.75 to a common fraction.

In order to change a common fraction to a decimal, divide the numerator by the denominator using the rule in Sec. 42 (see Example 4).

**Example 3.** Change the fraction $\frac{7}{8}$ to a decimal. Divide 7 by 8 to get 0.875.

In most cases the division process goes on without end. Then the common fraction cannot be changed into a decimal fraction exactly, which is actually never required in practice. The division is terminated when the quotient has as many decimal places as required in a given practical situation.

**Example 4.** It is required to divide 1 kilogram of coffee into three parts. The weight of each is 1/3 kg. To weigh this quantity, we have to express it in tenths of a kilogram (since there are no weights of one-third of a kilogram). Dividing 1 by 3, we get: $1:3 = 0.333 \ldots$ . The division can be continued endlessly with new threes appearing in the quotient. But small weights (say, less than 1 gram) are not indicated by ordinary scales; what is more, the coffee beans themselves weigh more than a gram each. Only hundredths of a kilogram (10 grams) are of practical interest in this case. And so we take $\frac{1}{3}$ kg $\approx$ 0.33 kg.

For greater accuracy, it is accepted usage to make allowance for the value of the last rejected digit. If it exceeds 5, the retained digit is increased by unity.

**Note.** Even when a common fraction can be expressed exactly as a decimal, this is not done in most cases. The division process is terminated as soon as the required degree of accuracy is attained.

**Example 5.** Change the fraction $\frac{7}{32}$ to a decimal. The exact value is 0.21875. Depending on the accuracy required,

the division process is terminated with the second, third, etc. digit of the quotient, and we take $\frac{7}{32} \approx 0.22$, $\frac{7}{32} \approx 0.219$, and so on.

### 45. Historical Survey of Fractions

The notion of a fraction could develop only after definite conceptions concerning whole numbers had been firmly established. Like the concept of an integer, the concept of a fraction developed gradually. The idea of "one half" * originated much before that of thirds or fourths, and the latter two appeared much earlier than fractions with other denominators. The first notion of a whole number evolved out of the process of counting, the first conception of fractions, out of the process of measuring (lengths, areas, weights, and so on). Many languages have traces of the historical connection between fractions and the existing system of measures. For example, in the Babylonian system of measures and money, 1 talent is composed of 60 minas, one mina making 60 shekels. Accordingly, Babylonian mathematics made extensive use of sexagesimal fractions (see Sec. 21). In the weight and monetary system of ancient Rome, 1 as consisted of 12 ounces (uncia); the Romans accordingly made use of duodecimal fractions. The fraction we call $\frac{1}{12}$ was called an "uncia" by the Romans even when it was used for measuring lengths or other quantities. The Romans called $\frac{1}{8}$ one and a half ounces, and so forth.

Our common fractions were widely used by the ancient Greeks and Hindus. The rules for handling fractions given by the Hindu scholar Brahmagupta (8th century) differ but slightly from our own rules. Our way of writing fractions coincides with the Hindu custom. True, the Hindus did not use a fraction bar. The Greeks wrote the denominator above the numerator, although other forms of notation were used more often. For example, they wrote (using other symbols, naturally) $35^{\kappa}$ (three fifths).

The Hindu symbolism for fractions and rules for handling fractions spread into the Muslim world in the 9th century

* In all languages, the concept "half" has a special name not connected with the word "two". Originally "half" meant one of two parts (which were not necessarily equal).

due to al-Khowarizmi (see Sec. 21), and thence to Western Europe in the 13th century thanks to the Italian merchant and scholar Leonardo of Pisa (also known as Fibonacci).

Besides common fractions, sexagesimal fractions were in use, especially in astronomy. The latter subsequently gave way to decimal fractions, which were first introduced by the celebrated Samarkand scholar al-Kashi (14th to 15th century). In Europe, decimal fractions were introduced by the Flemish mathematician and engineer (he was also a merchant) Simon Stevin (1548—1620).

### 46. Percentage

The expression "per cent" (from the Latin "per centum", "by the hundred") means a hundredth part. Symbolically, 1% stands for 0.01; 27% for 0.27; 100% for 1; 150% for 1.5, etc. (the symbol for percentage, %, is a distortion of the notation $c_{to}$, which is a contraction of the word "cento").

1% of a sum means 0.01 of it; to fulfil a plan means to complete 100% of it; whereas fulfillment by 150% would mean that one and a half quotas of the planned amount had been completed, and so forth.

To find the percentage expression of a given number, multiply the number by 100 (or, what is the same, move the decimal point two places to the right).

**Examples.** Expressed as a percentage, 2 is 200%; the number 0.357 is 35.7%, the number 1.753 is 175.3%.

To change a percent to a number, divide the percent by 100 (or, what is the same, move the decimal point two places to the left).

**Examples.** $13.5\% = 0.135$, $2.3\% = 0.023$, $145\% = 1.45$, $^2/_5\% = 0.4\% = 0.004$.

The three principal problems involving percentage are:

**Problem 1. Find the indicated percent of a given number.** (cf. Sec. 38, Rule 1). Multiply the number by the percent and divide by 100 (or, what is the same, move the decimal point two places to the left: in other words, the given number is multiplied by the fraction expressing the given percent).

**Example.** A planned quota in coal production is 2860 tons per day. A mine pledges to do 115% of the plan. How many tons of coal will it mine per day?

**Solution.** (1) $2860 \cdot 115 = 328{,}900$.

(2) $328{,}900 : 100 = 3289$ tons

(which is equivalent to $2860 \cdot 1.15 = 3289$).

**Problem 2. Find a number on the basis of a given percent** (cf. Sec. 38, Rule 2). The given quantity is divided by the percent and then multiplied by 100 (or the decimal point is moved two places to the right, which is to say the given number is divided by the fraction expressing the given percent).

**Example.** In processing sugar beets, 12.5% of the weight of the beets is granulated sugar. What quantity of beets has to be processed to produce 3000 centners of granulated sugar?

**Solution.** (1) $3000:12.5=240$. (2) $240\cdot100=24{,}000$ (centners) (which is tantamount to writing $3000:0.125=24{,}000$).

**Problem 3. Find what percent one number is of another** (cf. Sec. 38, Rule 3). Multiply the first number by 100 and divide by the second number.

**Example 1.** A new burning process for brick manufacture made it possible to increase the output of bricks per cubic metre of furnace from 1200 to 2300 bricks. What was the increase in brick output in percentage?

**Solution.**

(1) $2300-1200=1100$,

(2) $1100\cdot100=110{,}000$,

(3) $110{,}000:1200\approx91.67$.

Brick output increased by 91.67%.

**Example 2.** According to the seven-year plan, the petroleum output in the USSR was to reach 161 million tons in 1961. Actually, 166 million tons were produced. Give the fulfillment of the 1961 plan in percentage.

**Solution.**

(1) $166\cdot100=16{,}600$,

(2) $16{,}600:161\approx103.1$.

Petroleum output in 1961 was 103.1% of the planned amount.

**Note 1.** In all three types of problems, the sequence of operations can be changed (say, in the last problem, we could first divide and then multiply by 100).

**Note 2.** The example which follows is to serve as a warning against a mistake that is very frequently made.

It is required to find out the price of a metre of cloth prior to a price reduction if after a price reduction of 15% the price is 12 roubles per metre. Sometimes, 15% of 12 roubles is found, that is, $12\cdot0.15=1.8$. This is followed by the addition $12+1.8=13.8$, and it is taken that the old price was 13.8 roubles per metre. This is not so because the percent of reduction is established with respect to the earlier

prices, and 1.8 roubles is not 15% of 13.8 roubles but about 13% (see Problem 3). The correct solution is this: after the price reduction, the cloth cost $100\% - 15\% = 85\%$ of the earlier price. And so the old price (see Problem 2) was $12 : 0.85 = 14.12$ roubles per metre.

**Note 3.** In working percentage problems it is best to take advantage of the methods of approximate computations (see the sections which follow).

### 47. Approximate Calculations

The numbers we deal with in everyday affairs are of two kinds. Some state exact magnitudes, others give only approximate values. Thus, we have exact numbers and approximate numbers. We often take an approximate value in place of an exact value simply because the latter is not required. In many cases it is simply impossible to find a number exactly.

**Example 1.** The number of pages in a book is exact. This book has 423 pages.

**Example 2.** A hexagon has 9 diagonals, which is an exact number.

**Example 3.** A salesman weighs 50 grams of butter. 50 is an approximate number because the scales are not sensitive to an increase or decrease of 0.5 gram.

**Example 4.** The distance by railway from Moscow to Leningrad is 651 kilometres. The number 651 is an approximate number because our measuring instruments are not exact.

Operations involving approximate values yield approximate values. What is more, inexact digits may result from operations on the exact digits of the given numbers.

**Example 5.** In multiplying the approximate numbers 60.2 and 80.1 let us suppose that all the indicated digits are correct so that the true values can differ from the approximate ones only in hundredths, thousandths, etc. The product is 4822.02. Here, not only the hundredths and tenths digits but even the units digits may be incorrect. For example, suppose that the factors were obtained by rounding off (see Sec. 49) the exact numbers 60.25 and 80.14. Then the exact product is 4828.435, so that the units digit in the approximate product (2) differs from the exact figure (8) by 6 units.

If we know the degree of accuracy of the starting figures, the theory of approximate computations permits us (1) to estimate the degree of accuracy of the results prior to performing the operations, (2) to take the initial numbers

with the desired degree of accuracy so as to ensure the required accuracy of the result without involving the computor in extra needless computations, (3) to rationalize the very process of computation by omitting computations that do not affect the final exact figures of the result.

### 48. Notation of Approximate Numbers

In approximate calculations we distinguish between 2.4 and 2.40, between 0.02 and 0.0200 and so on. The notation 2.4 means that only the units and tenths are correct; the true value of the number may be, say, 2.43 or 2.38 (when the digit 8 is dropped, we round the preceding digit upwards; see Sec. 49). The notation 2.40 means that the hundredths are correct; the true number may be 2.403 or 2.398 but not 2.421 or 2.382.

This distinction also holds true for whole numbers. The notation 382 means that all digits are correct; now if there is any doubt about the last digit, the number is rounded off and is written as $38 \cdot 10$ and not 380. If we write 380 this means that the last digit (0) is true. If in the number 4720 only the first two digits are correct, then it must be written as $47 \cdot 10^2$; this number can also be written in the form $4.7 \cdot 10^3$, etc.

The *significant digits* of a number are all the correct digits (except for zeros) which stand at the beginning of the number. For example, in 0.00385 there are three significant digits; in the number 0.03085 there are four significant digits; in 2500 there are four, and in $2.5 \cdot 10^3$ there are two.

### 49. Rules for Rounding Off Numbers

In approximate computations it is frequently necessary to round off numbers (both approximate and exact), which means dropping one or more of the last digits. To ensure that the rounded number is as close as possible to the original number, use the following rules:

**Rule 1.** If the first of the discarded digits exceeds 5, then the last digit kept is increased by unity. The increase is also made when the first digit kept is equal to 5 and is

followed by one or more significant digits (for the case when the discarded 5 is not followed by any digits, see Rule 3 below).

**Example 1.** Rounding the number 27.874 to three significant digits, we write 27.9. The third digit 8 is increased to 9 since the first discarded digit, 7, exceeds 5. The number 27.9 is closer to the original number than the rounded (but not increased) number 27.8.

**Example 2.** Rounding the number 36.251 to the first decimal, we write 36.3. The tenths digit 2 is increased to 3 because the first discarded digit is equal to 5 and it is followed by one significant digit, 1. The number 36.3 is closer to the initial number (though only slightly) than the unincreased number 36.2.

**Rule 2.** If the first digit dropped is less than 5, no increase is made.

**Example 3.** Rounding 27.48 to the nearest whole number, we write 27. This number is closer to the given one than 28.

**Rule 3.** If the digit 5 is dropped, and no significant digits come after it, the rounding is done to the closest even number; that is, the last retained digit is left unchanged if it is even and is increased if it is odd. The reason for this rule is given below (see note).

**Example 4.** Rounding 0.0465 to the third decimal place, we write 0.046. We do not increase the last digit kept since it is even. The number 0.046 is just as close to the given one as is 0.047.

**Example 5.** Rounding 0.935 to the second decimal place, we write 0.94. The last retained digit 3 is increased because it is odd.

**Example 6.** Rounding the numbers

6.527, 0.456, 2.195, 1.450, 0.950, 4.851, 0.850, 0.05 to the first decimal place, we get

6 5, 0.5, 2.2, 1.4, 1.0, 4.9, 0.8, 0.0

**Note.** When applying Rule 3 to the rounding off of one number do not increase the accuracy of the rounding process (see Examples 4 and 5). However, if the process is performed repeatedly, there will be roughly just as many numbers with excess as with deficit. The mutual compensation of errors will ensure the highest possible accuracy of the result.

Rule 3 can be modified and made to apply to rounding off to the closest odd number. The accuracy will be the same but even digits are more convenient than odd digits.

## 50. Absolute and Relative Errors

The *absolute error* (or, simply, error) of an approximate number is the difference between the number and its exact value (the small number is subtracted from the greater).*

**Example 1.** There are 1284 employees in a given institution. Rounding this number off to 1300, we get an absolute error of $1300-1284=16$. Rounding off to 1280, we have an absolute error of $1284-1280=4$.

The *relative error* of an approximate number is the ratio (see Sec. 62) of the absolute error of the approximate number to the number itself.

**Example 2.** A school has a student body of 197. Rounding this number to 200, we obtain an absolute error of $200-197=3$. The relative error is equal to $\frac{3}{197}$ or, rounded, $\frac{3}{200}=1.5\%$.

In most cases it is impossible to determine the exact value of an approximate number and, hence, the exact value of the error. However, it is almost always possible to establish that the error (absolute or relative) does not exceed a certain number.

**Example 3.** A salesman weighs a watermelon on pan scales. The smallest of the set of weights is 50 grams. The result is 3600 grams. This is an approximate number. The exact weight of the watermelon is not known. However, the absolute error does not exceed 50 grams, and the relative error does not exceed $\frac{50}{3600}\approx 1.4\%$.

A number which definitely exceeds the absolute error (or, at worst, is equal to it) is called the *limiting absolute error*. A number which definitely exceeds the relative error (or, at worst, is equal to it) is called the *limiting relative error*.

In Example 3, we can take 50 grams for the limiting absolute error and 1.4% for the limiting relative error.

The magnitude of a limiting error is not quite definite. Thus, in Example 3 we can take 100, 150 and generally any number over 50 grams for the limiting absolute error. In practical cases the smallest possible value of the limiting

* In other words, if $a$ is an approximate number and $x$ its exact value, then the absolute error is the absolute value (Sec. 69) of the difference $a-x$. In some manuals, the absolute error is defined as the difference itself $a-x$ (or the difference $x-a$). This quantity can be positive or negative.

error is taken. When the exact magnitude of the error is known, this serves, at the same time, as the limiting error.

For every approximate number, we must know its limiting error (absolute or relative). When it is not directly indicated, it is assumed that the limiting absolute error represents one half of one unit of the last written digit. For instance, if we have the approximate number 4.78 without the limiting error indicated, then the limiting absolute error is assumed to be 0.005. With this convention, we can always do without indicating the limiting error of a number rounded off by the rules of Sec. 49.

The limiting absolute error is denoted by the Greek letter delta ($\Delta$); the limiting relative error, by the lower-case Greek delta ($\delta$). If an approximate number is denoted by $a$, then $\delta = \frac{\Delta}{a}$.

**Example 4.** A pencil is measured with a ruler calibrated in millimetres and yields a result of 17.9 cm. What is the limiting relative error of this measurement?

Here, $a = 17.9$ cm; we can take $\Delta$ to be equal to 0.1 cm, since it is not difficult to measure a pencil to within 1 mm, yet it will not be possible to reduce substantially the limiting error (with practice, it is possible to read 0.02 cm, and even 0.01 cm, on a good ruler but at the very edge of the ruler the discrepancy may be greater). The relative error is $\frac{0.1}{17.9}$. Rounding off, we get $\delta = \frac{0.1}{18} \approx 0.6\%$.

**Example 5.** A cylindrical piston is about 35 mm in diameter. To what degree of accuracy must a measurement be made with a micrometer so that the limiting relative error is 0.05%?

**Solution.** It is given that the limiting relative error must constitute 0.05% of 35 mm. Consequently (Sec. 46, Problem 1), the limiting absolute error is $\frac{35 \cdot 0.05}{100} = 0.0175$ (mm) or, rounding upwards, 0.02 (mm).

We can use formula $\delta = \frac{\Delta}{a}$. Substituting $a = 35$, $\delta = 0.0005$, we get $0.0005 = \frac{\Delta}{35}$. Thus,

$$\Delta = 35 \cdot 0.0005 = 0.0175 \text{ (mm)}$$

### 51. Preliminary Rounding Off in Addition and Subtraction

If the given numbers do not all end in the same digit place (order), round off before performing the addition or subtraction. In other words, retain only those digit places

that are good for all addends, the others being dropped as useless. For a small number of addends, all digits of the sum, except the last, will be correct. The last digit may not be quite exact. This inexactness can be reduced to a minimum if we take into account the digits of the next digit place *(extra digits)*.

**Example 1.** Find the sum of $25.3+0.442+2.741$.

Without rounding off the terms, we get 28.483. The last two digits are useless since there is a possible inaccuracy of several hundredths in the first addend. Rounding the sum to exact digits (that is, to tenths), we get 28.5. If we first round off to exact digits, then we readily get $25.3+0.4+2.7=28.4$. The tenths digit is less by 1. Taking the hundredths digits as well, we get $25.3+0.44+2.74=28.48$, which, rounded, is 28.5. The digit 5 is more reliable than 4, though it might very well be that the true figure is precisely 4. *

When using extra digits, arrange the computation as indicated in the accompanying scheme with the extra digits separated by a vertical line:

**Work:**

$$\begin{array}{rr|l} & 25.3 & \\ & 0.4 & 4 \\ + & 2.7 & 4 \\ \hline & 28.5 & \end{array}$$

**Example 2.** Find the sum of $52.861+0.2563+8.1+57.35+0.0087$.

Without using any extra digits (we retain only rounded tenths; see rules for rounding, Sec. 49), we get 118.7. With extra digits, we have 118.6. In the latter result, the tenths digit may prove to be incorrect due to the inaccuracy of the third addend; a 5 may appear in place of 6 (if the third addend has been rounded off from 8.06). But 6 is much more reliable. At any rate 7 cannot be correct. Extra digits yield an improvement, but only a slight one. Compare the two schemes: on the left without extra digits, on the right using

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* If we assume the first addend to be 25.26 rounded, then the sum to hundredths would be 28.44, or approximately 28.4. However, if 25.3 is the rounded number 25.27 or 25.28, etc., then the sum will be 28.5 after rounding.

extra digits:

| Work: | 52.9 | 52.8 | 6 |
|---|---|---|---|
| | 0.3 | 0.2 | 6 |
| | + 8.1 | 8.1 | |
| | 57.4 | + 57.3 | 5 |
| | 118.7 | 0.0 | 1 |
| | | 118.6 | |

### 52. Error of a Sum and a Difference

The limiting absolute error of a sum is equal to the sum of the limiting absolute errors of the separate addends.

**Example 1.** The approximate numbers 265 and 32 are added. Let the limiting error of the former be 5, of the latter, 1. Then the limiting error of the sum is equal to $5+1=6$. Thus, if the true value of the former number is 270, of the latter, 33, then the approximate sum $(265+32=297)$ is 6 less than the true value $(270+33=303)$.

**Example 2.** Find the sum of the approximate numbers $0.0909+0.0833+0.0769+0.0714+0.0667+0.0625+0.0588+0.0556+0.0526$.

Addition yields 0.6187. The limiting error of each addend is 0.00005; the limiting error of the sum is $0.00005\times 9=0.00045$. Thus, there is a possible error of up to 5 units in the last (fourth) decimal place of the sum, and so we round off the sum to the third decimal place (thousandths). This yields 0.619; where all the decimal places are correct.

**Note.** A large number of addends usually makes for a balancing of errors; for this reason, only in exceptional cases does the true error of a sum coincide with the limiting error or come close to it. That these cases are rare is seen from Example 2 where we had 9 addends. The true value of each of them can differ in the fifth decimal place from the given approximate value by 1, 2, 3, 4 or even 5 units either way. For example, the first addend may be greater than its true value by 4 units of the fifth decimal place, the second by two units, the third, less by one unit, etc. Calculations show that the number of all possible cases of the distribution of errors is about 1000 million. But only in two cases can the error of the sum reach the limiting error of 0.00045. This occurs when (1) the true value of each addend exceeds the approximate value by 0.00005, and (2) the true value of each addend is less than the approximate value by 0.00005. Thus, the cases when the error of a sum coincides with the

limiting error constitute only 0.0000002% of all possible cases.

Further calculations show that cases when the error of a sum of nine addends can exceed three units of the last decimal place are also very rare. They amount to only 0.07% of all possible cases. An error can exceed two units of the last decimal place in 2% of all possible cases, one unit, in roughly 25%. In the remaining 75% of the cases, the error of nine addends does not exceed one unit of the last decimal place.

**Example 3.** Assuming the addends of Example 2 to be exact numbers, * let us round them off to thousandths and add. The limiting error of the sum will be $9 \cdot 0.0005 = 0.0045$. Yet we have

$$0.091 + 0.083 + 0.077 + 0.071 + 0.067 + 0.062 + 0.059 + 0.056 + 0.053 = 0.619$$

That is to say, the approximate sum differs from the true sum by 0.0003, which is a third of a unit of the last decimal place of the approximate numbers. All three decimals of the approximate sum are correct, although theoretically the last decimal might be glaringly inexact.

Let us round off to hundredths in our addends. Now the limiting error of the sum is $9 \cdot 0.005 = 0.045$. Yet we get $0.09 + 0.08 + 0.08 + 0.07 + 0.07 + 0.06 + 0.06 + 0.06 + 0.05 = 0.62$. The true error comes out to only 0.0013, which is $\frac{1}{8}$ of a unit of the last decimal of the approximate numbers.

The limiting absolute error of a difference is equal to the sum of the limiting absolute errors of the minuend and the subtrahend.

**Example 4.** Let the limiting error of the approximate minuend 85 be 2, and the limiting error of the subtrahend 32 be 3. The limiting error of the difference $85 - 32 = 53$ is $2 + 3 = 5$. Indeed, the true values of the minuend and subtrahend may be equal to $85 + 2 = 87$ and $32 - 3 = 29$. Then the true difference is $87 - 29 = 58$. It differs from the approximate difference 53 by 5.

---

* These addends are obtained by changing the common fractions $\frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \ldots, \frac{1}{19}$ to decimals to within the fourth decimal place. The reader can take other numbers at random.

The limiting relative error of a sum and a difference can easily be found by first computing the limiting absolute error (see Sec. 50).

The limiting relative error of a sum (but not of a difference) lies between the smallest and largest of the relative errors of the addends. If all the addends have the same (or roughly the same) limiting relative error, then the sum also has the same (or roughly the same) limiting relative error. In other words, in this case the accuracy of the sum (expressed as a percentage) is not inferior to the accuracy of the addends. For a large number of addends, the sum (as a rule) is much more accurate than the addends (for the reason explained in the note of Example 2).

**Example 5.** In each addend of the sum $24.4+25.2+24.7=74.3$ the limiting relative error is approximately the same, namely, $0.05:25=0.2\%$. It is the same for the sum as well. Here, the limiting absolute error is equal to 0.15, the relative error $0.15:74.3 \approx 0.15:75=0.2\%$.

In contradistinction to the sum, the difference between two approximate numbers may be less exact than the minuend and subtrahend. The "loss of accuracy" is particularly great when the minuend and subtrahend differ only slightly.

**Example 6.** Measurements of the outer and inner diameters of a thin-walled pipe yielded 28.7 mm and 28.3 mm, respectively. Using these figures, we find the wall thickness: $\frac{1}{2}\cdot(28.7-28.3)=0.2$ (mm). The limiting relative error of the minuend (28.7) and the subtrahend (28.3) is the same: $\delta=0.2\%$. The limiting relative error of the difference 0.4 (and also of half the difference, 0.2) comes out to 25%.

If follows, from the foregoing, that whenever possible one should avoid computing a desired quantity by subtracting nearly equal numbers. Cf. Sec. 90, Example 9.

### 53. Errors in a Product

The *limiting relative error of a product* is approximately equal to the sum of the limiting relative errors of the factors. (For the exact value of the limiting error see note of Example 1).

**Example 1.** Two approximate numbers, 50 and 20, are multiplied together. Let the limiting relative error of the first factor be 0.4%, of the second factor, 0.5%. Then the limiting relative error of the product $50\times20=1000$ is appro-

ximately 0,9%. Indeed, the limiting absolute error of the first factor is $50 \cdot 0.004 = 0.2$, of the second, $20 \cdot 0.005 = 0.1$. Therefore, the true value of the product does not exceed $(50 + 0.2) \cdot (20 + 0.1) = 1009.02$, and is not less than $(50 - 0.2) \times (20 - 0.1) = 991.02$. If the true value of the product is 1009.02, then the error of the product is equal to $1009.02 - 1000 = 9.02$ and if it is 991.02, then the error of the product is $1000 - 991.02 = 8.98$. These two cases are the most unfavourable ones. Hence the limiting absolute error of the product is 9.02. The limiting relative error is equal to $9.02 : 1000 = 0.902\%$, which is approximately 0.9%.

**Note.** Denote the limiting relative error of a product by the letter $\delta$, the limiting relative error of the factors by $\delta_1$ and $\delta_2$ (in Example 1, $\delta_1 = 0.004$, $\delta_2 = 0.005$, $\delta = 0.00902$).

Our rule (for two factors) then looks like this:

$$\delta \approx \delta_1 + \delta_2$$

The exact expression of $\delta$ is

$$\delta = \delta_1 + \delta_2 + \delta_1 \delta_2$$

That is, the limiting relative error of a product is always greater than the sum of the limiting relative errors of the factors; it exceeds this sum by the product of the relative errors of the factors. The excess is ordinarily so small that it can be ignored. Taking Example 1, we have $\delta = 0.004 + 0.005 + 0.004 \cdot 0.005 = 0.00902$. The excess here is $0.00902 - 0.009 = 0.00002$, which is about 0.2% of the approximate value of the limiting relative error. This excess is so small that it can be disregarded.

**Example 2.** Suppose the approximate numbers 53.2 and 25.0 are multiplied together. The limiting absolute error of each is 0.05. Therefore, $\delta_1 = 0.05 : 53.2 = 0.0009$, $\delta_2 = 0.05 : 25.0 = 0.002$. The limiting relative error of the product $53.2 \cdot 25.0 = 1330$ is approximately equal to $0.0009 + 0.0020 = 0.0029$. The quantity $\delta_1 \delta_2 = 0.0009 \cdot 0.002 = 0.0000018$ is so small that it is meaningless to take it into account. The limiting absolute error of the product 1330 is $1330 \cdot 0.0029 \approx 4$, so that the last digit of the product (zero) may be incorrect.

**Example 3.** Find the volume of a room, given the measurements: length 4.57 m, width 3.37 m, height 3.18 m (the limiting absolute errors are 0.005 m). Multiplying these numbers together we find the volume to be 48.974862 $m^3$. But only two digits are definitely correct here, the third may already have a slight error. Indeed, the limiting rela-

tive errors of the factors are:

$$\delta_1 = 0.005:4.57 \approx 0.0011, \quad \delta_2 = 0.005:3.37 \approx 0.0015,$$
$$\delta_3 = 0.005:3.18 \approx 0.0016$$

The limiting relative error of the product is

$$\delta = 0.0011 + 0.0015 + 0.0016 = 0.0042$$

The limiting absolute error of the product $\Delta \approx 49.0 \cdot 0.0042 \approx 0.21$. Thus the third significant digit of the product is unreliable. Hence we take the volume of the room to be 19.0 m³.

### 54. Counting Exact Digits in Multiplication

The error in a product may be estimated more simply (more crudely, true) than by the procedure given in Sec. 53. This estimate is based on the following rule.

Let two approximate numbers be multiplied together and let each have $k$ significant digits. Then the $(k-1)$st digit of the product is definitely correct, while the $k$th digit may not be quite exact. However, the error of a product does not exceed $5^1/_2$ units of the $k$th digit and only in exceptional cases is close to this limit. Now if the first digits of the factors in a product yield a number exceeding ten (either taking into account or disregarding the effect of the subsequent digits), the error of the product does not exceed one unit of the $k$th digit.

**Example 1.** Let us multiply together the approximate numbers 2.45 and 1.22, each of which has three significant digits. In the product, 2.9890, the first two digits are surely correct. The third digit may not be quite exact. For the given values of the factors, the limiting absolute error of the product (it may be found, as was done in Example 1, Sec. 53) constitutes 1.8 units of the third digit (or 0.0018); as a rule, the true error will be still less. Therefore the third digit should be retained and there is no sense in keeping the fourth digit. Rounding off, we have $2.45 \cdot 1.22 \approx 2.99$.

**Example 2.** Multiply the approximate numbers $46.5 \times 2.82$. In the product, 131.130, the first two digits are definitely correct. Since the first digits of the factors (with account taken of the subsequent digits) yield 13 in the product (the first two digits of the number 131.130), it follows that the error of the product definitely does not exceed unity. In this case, the limiting absolute error of the product is only 0.37;

the true error will, as a rule, be less. So the third digit must be retained. It is advisable to retain the fourth digit (which is not quite exact) as an extra digit only when other operations are to follow.

When multiplying together three, four or more approximate numbers, the limiting error increases proportionally (that is to say, it increases over that given above by $1\frac{1}{2}$, two, etc. times). However, in most cases, the true error for a small number of factors remains within the same limits (due to a compensation of errors; cf. Sec. 52).

## Practical Advice

1. If approximate numbers with the same number of significant digits are being multiplied together, retain the same number of significant digits in the product. The last retained digit will be in doubt.

2. If some of the factors have more significant digits than others, then, prior to multiplication, round off the longer numbers retaining as many digits as the least exact factor, or one more digit (as an extra digit). There is no sense in keeping any more digits.

3. If it is required that the product of two numbers have a prescribed number of reliable digits, then the number of exact digits in each factor (found via computation or measurement) must be one more. If the number of factors is greater than two but less than ten, then the number of exact digits in each factor must, for complete assurance, be two units more than the required number of exact digits. In most practical situations only one extra digit is quite sufficient.

To verify these conclusions, let us consider an example in which we already know the exact values of the approximate numbers being multiplied.

**Example 3.** Change the product $\frac{1}{3}\cdot\frac{1}{7}\cdot\frac{1}{11}\cdot\frac{1}{13}=\frac{1}{3003}$ to a decimal. Taking 4 significant digits, we get 0.0003330. Now suppose all we know are the approximate values of the factors (the reader is advised to take any other factors):

$$\frac{1}{3}=0.33333,\quad \frac{1}{7}=0.14286,\quad \frac{1}{11}=0.09091,\quad \frac{1}{13}=0.07692$$

and it is required to find the product to two significant digits. To be on the safe side we have to take all factors to

four significant digits, that is, we multiply $0.3333 \cdot 0.1429 \times 0.09091 \cdot 0.07692$

(1) We find $0.3333 \cdot 0.1429 = 0.04762857$.

Retaining four significant digits, we get 0.04763.

(2) Perform the following multiplication:

$$0.04763 \cdot 0.0909 = 0.0043300433$$

(3) Retaining four significant digits (three would have sufficed) and performing the last multiplication, we get

$$0.004330 \cdot 0.07692 = 0.0003331$$

The first two significant digits are definitely correct, so that the desired number is 0.00033. We cannot, beforehand, be certain of the correctness of the third significant digit. It proves correct however. The fourth significant digit is not quite exact, but the error does not exceed unity of the corresponding decimal place.

If we carry our answer to six decimal places, then we cannot beforehand be certain of the fifth digit. Actually, however, even the sixth digit is correct. Namely,

(1) $0.333 \cdot 0.143 = 0.047619$,

(2) $0.0476 \cdot 0.0909 = 0.00432684$,

(3) $0.00433 \cdot 0.0769 = 0.000333$

If we carry the computation to five decimals, then for the product we have 0.00032, i. e., the error is 1.3 units of the fifth decimal place.

### 55. Short-Cut Multiplication

When applying the rules of multiplication of exact numbers to approximate numbers we waste time and effort in the computation of digits that will be dropped at a later stage. The computational procedure can be made more efficient if we are guided by the following rules.

(1) Start the multiplication with the higher digit places of the multiplier (not the lower digit places); when multiplying the multiplicand by the highest digit place of the multiplier, carry out the multiplication completely.

(2) Before multiplication by the next place of the multiplier, cross-line the last digit in the multiplicand; multiply using a shortened multiplicand, but add to the result the rounded product of the given place of the multiplier by the discarded digit of the multiplicand.

(3) Before multiplying by the third (from the beginning) place of the multiplier, cross-line one more digit of the multiplicand (the second from the end); perform the multiplication by the remaining digits of the multiplier and take into account the effect of the digit that was just discarded and so on.

(4) The resulting products are arranged so that all the lower digit-orders are aligned one under another.

(5) There are special rules for setting the decimal point in the product, but it is a good practical rule to make a rough preliminary guess about the magnitude of the product. To avoid errors, it is advisable to cross-line each digit of the multiplier that has been used.

**Example 1.** Multiply the approximate numbers $6.7428 \cdot 23.25$. Equalize the number of significant digits: drop 8 in the first factor and replace 2 by 3. Compute by the accompanying scheme in the following sequence.

**Work:**

$$\begin{array}{r} 6.743 \\ \times\ 23.25 \\ \hline 13486 \\ 2023 \\ +\ \ 135 \\ \ \ \ 34 \\ \hline 156.78 \end{array}$$

(1) Disregard the decimal points and multiply 6743 by 2; write out the result in full: 13486; as usual, begin to multiply with $2 \cdot 3 = 6$ (write the 6 under the lowest digit-orders of the factors).

(2) Cross-line the digit of the multiplier that was used (2) and the last digit of the multiplicand (3); multiply the next digit of the multiplier (3) by the shortened multiplicand 674, noting that the cross-lined digit 3 would have given $3 \cdot 3 = 9$ in the product; and so add 1 to the product (from the very beginning of the multiplication, $3 \cdot 4 = 12$; $12 + 1 = 13$; write 3 and carry the 1). The lowest order of the product (3) is written under the lowest order of the preceding product (6).

(3) Cross out the second digit of the multiplier from the beginning and the second digit of the multiplicand from the end, multiply the third digit of the multiplier (2) by the shortened multiplicand 67, noting that multiplication of this digit of the multiplier by the discarded digit of the multiplicand would have yielded 8, and so we add 1 to the product.

(4) Finally, crossing out 2 in the multiplier and 7 in the multiplicand, multiply 5 by 6, noting first that $5 \cdot 7 = 35$, so that to the product $5 \cdot 6 = 30$ we add 4 (which is better than 3 since we would have had to multiply not only the digit 7 but also the discarded digits that follow it).

(5) Add all the products thus obtained to get 15,678.

In order to set the decimal point, we round the factors crudely taking, say, 6 in place of the first and 20 instead of the second. This gives us a rough product of 120, which

means that the integral part of our answer is a three-digit number; this means we must point off the first three digits and take 156.78 and not 15.678 or 1567.8. This answer is correct only to the first four digits. We use the last digit (which may contain an error of up to three units) to round off the result to 156.8.

**Example 2.** 674.3·232.5. The multiplication is performed as in the preceding example. We get 15,678 and to point off the decimal place we carry out the rough computation 600·200 = 120,000, which is a six-digit number. Since the integral part of our answer must contain six digits, and our number, 15,678, contains five, we annex a zero on the right; the decimal point lies outside the figures we obtained, i.e., the result of our multiplication yields the whole number 156,780. Since the last digit (zero) is definitely false, we write the answer as 15,678·10 or $1568 \cdot 10^2$ (see Sec. 48).

### 56. Division of Approximate Numbers

**Rule 1.** The limiting relative error of a quotient is approximately equal to the sum of the limiting relative errors of the dividend and divisor (cf. Sec. 53).

**Example 1.** Divide the approximate number 50.0 by the approximate number 20.0. The limiting error of the dividend and divisor is 0.05. Then the limiting relative error of the dividend is $\frac{0.05}{50.0} = 0.1\%$, and the limiting relative error of the divisor is $\frac{0.05}{20.0} = 0.25\%$. The limiting relative error of the quotient of 50.0 by 20.0 = 2.50 must be approximately $0.1\% + 0.25\% = 0.35\%$.

Indeed, the true value of the quotient does not exceed (50.0 + 0.05):(20.0 − 0.05) = 2.50877 and is not less than (50.0 − 0.05):(20.0 + 0.05) = 2.49127. If the true value of the quotient is 2.50877, then the absolute error comes out to 2.50877 − 2.50 = 0.00877. But if the true value is 2.49127, then the absolute error comes to 2.50 − 2.49127 = 0.00873. The foregoing cases are the most unfavourable. Hence, the limiting relative error is 0.00877:2.50 = 0.00351, or, approximately, 0.35%.

**Note.** The exact limiting relative error always exceeds the approximate error as computed by Rule 1. The percent of excess is roughly equal to the limiting relative error of the

divisor. In our example, the excess is 0.00001, which is 0.29% of 0.0035, whereas the relative error of the divisor is 0.25%.

**Example 2.** Find the limiting absolute error of the quotient of 2.81 by 0.571.

**Solution.** The limiting relative error of the dividend is $0.005:2.81=0.2\%$; of the divisor, $0.0005:0.571=0.1\%$; of the quotient, $0.2\%+0.1\%=0.3\%$. The limiting absolute error of the quotient is approximately equal to $\frac{2.81}{0.571}\times 0.003 = 0.015$. Hence, we are already uncertain about the third significant digit in the quotient $2.81:0.571=4.92$.

A simpler but more crude estimate of the accuracy of the quotient is based on a count of the exact digits (cf. Sec. 54). The estimate is as follows.

**Rule 2.** Suppose the dividend and the divisor each have $k$ significant digits. Then the absolute error of the quotient is at worst close to 1.05 units of the $(k-1)$st place (this value will never be attained).

As we see, the limiting error of a quotient is theoretically twice the limiting error of a product (see Sec. 54). Actually, however, the error of a quotient exceeds by 5 units the $k$th digit only in exceptional cases (once in a thousand). Therefore, *one should take as many significant digits in the quotient as there are in the dividend and divisor.*

If one of the given numbers (dividend or divisor) has more significant digits than the other, then drop all the superfluous digits or retain only the first one (as an extra digit).

If it is required that the quotient have a prescribed number of correct digits, take one extra significant digit in the dividend and divisor.

### 57. Short Division

In order to avoid superfluous computations, carry out the division of approximate numbers as follows.

(1) Disregarding the position of the decimal points, obtain the first digit of the quotient in the same manner as for whole numbers. If the significant digits of the dividend form a number that exceeds the significant digits of the divisor (both are regarded as whole numbers), then the first digit of the quotient is multiplied by the entire divisor. Otherwise cross out the last digit in the divisor and multiply by the shortened divisor, but take into account the effect of the

discarded digit. Thus, if we divide 2262 by 7646, the first digit of the quotient 2 (22:7 = 3 with a remainder, but 3 is not suitable and so we take 2). It is then multiplied into 764 and 1 is added to the result (this is the first digit of the product $2 \cdot 6 = 12$). This is done immediately upon multiplication by the last digit of the shortened divisor.

(2) The result of multiplying the first digit of the quotient by the divisor (or by the shortened divisor) is written under the dividend, aligning digit place under digit place. Then we find the remainder.

(3) Instead of bringing down a zero to the remainder, we shorten the divisor by cross-lining the last digit (if the shortening has already been done, then drop the last of the remaining digits). Choosing the second digit of the quotient, multiply it by the shortened divisor taking into account the digit just discarded.

(4) Write the result of the multiplication under the first remainder and align the digit places. We then find the second remainder.

(5) Instead of bringing down a zero we shorten the divisor by one more digit, etc.

6) Having obtained the quotient, we set off the decimal point by a rough estimate.

**Example 1.** 58.83:9.658.

**Work:**

58.83 | 9.658
57.95 | 6.092
88
86
2
2

(1) Since 5883 is less than 9658, cross out the last digit of the divisor, 8, from the very start. The first digit of the quotient is 6. Multiply by 965, noting that the discarded digit gives 5 units ($6 \cdot 8 = 48$; drop 8 and round 4 to 5).

(2) Write the product 5795 under the dividend, digit place under digit place. The remainder is 88.

(3) Cross out the second from the last digit of the divisor, 5. The shortened divisor, 96, is not contained even once in the dividend 88; put a zero in the quotient, * do not multiply.

(4) There is no need to find the second remainder.

(5) Cross out one more digit of the divisor, 6. The shortened divisor, 9, is contained in the remainder 9 times, and so the third digit of the quotient is 9. Multiplying by the shortened divisor with account taken of the effect of the

* Take careful note of this: a frequent mistake is made by not writing the zero and dropping the next digit of the divisor.

cross-lined digit, we have 86 and a remainder of 2. This is not the end of the operation. Dropping the last remaining digit, but taking into account its effect on the result, we find in the quotient another digit 2 ($2 \cdot 9 = 18$, the 8 is dropped and the 1 is rounded to 2). The last digit is obtained in simplest fashion by mentally bringing down zero to the last remainder 2; this yields $20:9 \approx 2$.

(6) A rough calculation gives the position of the decimal point. In the dividend and divisor we retain only the integral parts; it is clear that $58:9 \approx 6$, that is, the integral part of the quotient is a one-digit number. The result is therefore equal to 6.092, and not 60.92 or 6092, etc.

All digits of the answer are correct.

**Example 2.** 98.10:0.3216.

**Work:**

| | |
|---|---|
| 98.10 | 0.3216 |
| 96.48 | 305.0 |
| 162 | |
| 161 | |
| 1 | |

(1) 9810 is greater than 3216. Multiply the first digit of the quotient, 3, by 3216 to get 9648.

(2) The remainder is 162.

(3) Cross-line the last digit, 6, of the divisor. The shortened divisor, 321, does not go even once into the remainder; the second digit of the result is zero.

(4) and (5) Cross-line another digit of the divisor, 1; the remainder 162 is divided by the shortened divisor 32; for the third digit of the quotient we have 5. Multiply it by 32 and take into account the effect of the discarded digit of the divisor to get 161. Subtract it from the remainder to get 1. Cross-line the digit 2 in the divisor. The shortened divisor 3 does not go into the remainder 1, and so the last digit of the quotient is zero.

(6) Set the decimal point on the basis of a rough rounding off of the given figures: taking 100 in place of 98.10 and 0.3 in place of 0.3216, we get $100:0.3 \approx 300$; hence the integral part of the quotient is a three-digit number. The quotient therefore is 305.0.

### 68. Involution and Evolution of Approximate Numbers

Raising a number to an integral power (involution) is simply iterated multiplication, and therefore everything stated in Secs. 54, 55 holds true. When raising a number to a small power, the result has as many correct digits as the original number or it contains a slight error in the last di-

git. If the degree is large, then the accumulation of small errors may affect higher digit places.

When extracting a root (evolution), the result has at least as many correct digits as the radicand. Thus, taking the square root of the approximate number 40.00, we can obtain four correct digits ($\sqrt{40.00} \approx 6.324$). *

The method for finding the square root of a number that is frequently taught in school is cumbersome and hard to remember, and its theoretical justification remains rather obscure to most students. Below we give a simple and easy-to-remember procedure for taking the square root of a number (to any required degree of accuracy). This method was described by the ancient Greek scholar Heron roughly two thousand years ago (Heron used common fractions, but we, naturally, will use decimals). The same method can be used to extract third and higher roots (see Sec. 58a below).

**Rule for taking the square root.** To find the square roots, make a reasonable guess as a first approximation and do as follows.

(1) Divide the radicand by the first approximation of the root; if it turns out that the quotient differs from the first approximation by a quantity that does not exceed the permissible error, the root is found.

(2) Otherwise we find the arithmetic mean (Sec. 59) of the divisor and quotient. This arithmetic mean yields a considerably more exact value (second approximation) of the root. If the choice of the first approximation has been felicitous, the second approximation yields three correct digits, ordinarily not less than four correct digits. Generally speaking, the number of correct digits is doubled in each new approximation.

(3) Subject the second approximation to the same test as the first: divide the radicand by the second approximation. If the accuracy of the result is not sufficient, then find the third approximation and proceed as before, etc.

**Note 1.** The foregoing method is "not afraid" of mistakes since it automatically corrects any error made in the preceding stage. The sole drawback is a slowing down of the computation process.

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* If we employ the procedure of extracting a square root usually studied in school, then we will have to annex four zeros to the radicand writing 40.000000, in order to obtain 6.324. The annexed zeros will be false digits, but the corresponding digits of the answer will be correct. The result will remain the same if in place of four zeros we annex four arbitrary numbers.

**Example 1.** $\sqrt{40.00}$. The radicand has four significant digits. There is no sence in finding more than four digits of the root, and so we will take four.

For the first approximation we need a number between 6 and 7 (since $6^2=36$ is less than the radicand, and $7^2=49$ exceeds it). Within these limits we can take any number, but if we want to save time and effort, we have to take a number less than 6.5 (since the radicand is much closer to $6^2$ than to $7^2$). Let us take 6.4 (we could have taken 6.3 or 6.2, but 6.1 is no good because 6.1 is too close to 6). Now do as follows.

(1) Divide the radicand 40.00 by the first approximation 6.4. We have $40.00:6.4=6.25$. It is already evident that the second digit of the quotient 6.25 differs from the dividend 6.4. This accuracy does not suffice.

(2) For the second approximation we take the arithmetic mean of the dividend 6.40 and the quotient 6.25 to get $(6.40+6.25):2=6.325$. We may expect that if not all four digits are correct in this second approximation, then at least the first three are.

(3) To check, divide the radicand 40.00 by the second approximation 6.325 (carry the division to the fourth digit): $40.00:6.325\approx 6.324$. The quotient, 6.324, differs from the divisor, 6.325 by only one unit in the third decimal place, which means that the root has been found (to the required accuracy).

Indeed, squaring 6.324 (that is, multiplying that number by itself) we get a number less than the product of $6.324\times 6.325$, which is 40.00 (approximately). Now if we square 6.325, we get a number greater than $6.325\cdot 6.324\approx 40.00$. Hence, the desired square root lies between 6.324 and 6.325, which means that it differs from 6.324 (or from 6.325) by less than one unit in the third decimal place: $\sqrt{40.00}\approx 6.324$ (all four digits are correct).

**Example 2.** $\sqrt{23.5}$. The desired root lies between 4 and 5 and is much closer to 5 than to 4 (since 23.5 is much closer to 25 than to 16). For the first approximation let us take the round number 5.0.

(1) Divide the radicand 23.5 by the first approximation 5.00 (carrying the quotient to the third digit): $23.5:5.0=4.70$.

(2) For the second approximation we take the arithmetic mean $(5.00+4.70):2=4.85$. We may expect that all three digits are correct.

(3) To check, divide the second approximation 4.85 into the radicand 23.5 to get $23.5:4.85 \approx 4.85$. Since the quotient is equal (to within the second decimal place) to the divisor, the root is found (to the highest possible degree of accuracy): $\sqrt{23.5} \approx 4.85$.

**Note 2.** If the radicand is a decimal fraction with one significant digit or zero in the integral part, then to find the first approximation it is advisable to move the decimal point to the right two, four, six, etc. digits so that the integral part has a small number of places. Then proceed as in Examples 1 and 2, and in the answer move the decimal point back one, two, three, etc. digits. The procedure is similar when the radicand has a multidigit whole-number portion; but then the decimal point is first moved to the left two, four, six, etc. digits.

In the radicand, the decimal point can only be moved an *even number of digits.*

**Example 3.** $\sqrt{0.008732}$. Move the decimal point 4 digits to the right: 87.32. In choosing the first approximation, we will take into account only the integral part. Let us take, say, 9.3.

(1) Divide 9.3 into 87.32. Carrying the division to the fourth significant digit, we get $87.32:9.3 \approx 9.389$.

(2) Find the arithmetic mean $(9.300+9.389):2 \approx 9.344$.

(3) To check, perform the division $87.32:9.344 \approx 9.345$. In either of the two numbers 9.344 and 9.345 all four digits are correct (the first number yields a deficit, the second, an excess).

(4) Since, at the beginning, we moved the decimal point rightwards 4 places, we now move it to the left (back) 2 places, and we have

$$\sqrt{0.008732} \approx 0.09344$$

**Example 4.** $\sqrt{8732000}$. Move the decimal point to the left 6 digits to get 8.732 (if we move it 4 digits, we get 873.2 and not 87.32 as in the previous example!). Take 3 as the first approximation.

(1) $8.732:3 = 2.911$.

(2) $(3.000+2.911):2 = 2.955$.

From the first operation it is clear that there were two correct digits in the first approximation (3.000). We therefore expect to have 4 correct digits in the second approximation. A check confirms this.

(3) Since we started by moving the decimal point 6 places to the left, we now come back three places: $\sqrt{8732000} \approx 2955$.

### 58a. Rule for Extracting a Cube Root

To extract the cube root of a number, make a plausible first estimate and proceed as follows.

(1) Divide by the first estimate (cf. rule of Sec. 58) twice: the dividend is first the radicand and then the number obtained by the first division. If the quotient (obtained by the second division) differs from the first estimate (approximation), that is to say, from the divisor, by a quantity not exceeding the permissible error, then the process is complete.

(2) Otherwise, average three numbers, namely the quotient (of two divisions) and the divisor taken twice (see Example 1 for an illustration of this second operation). We get a second approximation; which, if the first estimate was plausible enough, is correct to three digits, the fourth digit at worst requires a correction by unity.

(3) The second estimate can be tested in the same manner as the first, but this is a tiring procedure.

**Example 1.** $\sqrt[3]{785.0}$. The desired root lies between 9 and 10. Take 9.2 as a first estimate (since the radicand is roughly four times closer to $9^3$ than to $10^3$).

(1) Divide 9.2 into the radicand 785.0 and then into the quotient of 785.0:9.2. Instead, we can divide 785 by $9.2^2$ to get 84.64. This yields

$$785.0:9.2:9.2 = 785.0:84.64 \approx 9.275$$

We see that the first estimate yields two correct digits. The best way to make the second estimate is to note that the radicand 785.0 is a product of three unequal factors: $785.0 = 9.2 \times 9.2 \times 9.275$, whereas we have to represent it as the product of three equal factors: $785.0 = x \cdot x \cdot x$ (where $x = \sqrt[3]{785.0}$). It is natural to assume that each of these equal factors should, approximately, be equal to the average of the factors 9.2, 9.2 and 9.275.

(2) Thus, for the second estimate we take the average $(9.275 + 9.200 + 9.200):3 = 9.225$. Compute by the short-cut method (see Sec. 60).

(3) For a check, divide the radicand 785.0 by the second estimate 9.225 and then divide the result again by 9.225 (or

divide the radicand by $9.225^2 \approx 85.09$). We get 9.225 (if an extra digit is not retained in the computations, we get 9.224):

$$\sqrt[3]{785.0} \approx 9.225 \text{ (correct to 4 figures)}$$

**Note.** In making the first estimate it is sometimes advantageous to move the decimal point in the radicand to the right (or to the left) 3, 6, 9, etc. places (cf. Sec. 58, Note 2). In the final result, move the decimal point back 1, 2, 3, etc. places. The decimal point may be moved only by as many digits as is divisible by 3.

**Example 2.** $\sqrt[3]{1835.10}$. In the radicand, 18,350, move the decimal point three places to the left to get 18.35. This number is roughly midway between $2^3=8$ and $3^3=27$. So for the first approximation we take 2.5.

(1) Divide twice by 2.5 or, what is the same, once by $2.5^2$. We get $18.35:2.5:2.5=18.35:6.25 \approx 2.94$.

We see that in this first approximation only one digit is correct. Thus, we must expect that in the second approximation there will only be two correct digits. Therefore, in the next operation we carry the answer only to two places.

(2) For the second estimate take the average $(2.5+2.5+2.9):3 \approx 2.6$.

(3) To make the result more precise we divide twice by 2.6 to get

$$18.35:2.6:2.6=18.35:6.76 \approx 2.715$$

We see that the second estimate furnished two correct digits, and so the third most likely will yield 4 correct digits.

(4) For the third estimate, average

$$(2.715+2.600+2.600):3=2.638$$

A check (which we omit) would show that the result is correct to four significant digits:

$$\sqrt[3]{1835.10} \approx 26.38$$

### 59. Mean Quantities

If we have a sequence of quantities (numbers), any one between the smallest and greatest is a mean. The most frequently used mean quantities are the arithmetic mean and the geometric mean.

The *arithmetic mean* (arithmetic average, or, simply, average) is obtained by adding the given quantities and dividing the sum by the number of quantities:

$$\text{a. m.} = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

($a_1$, $a_2$, ..., $a_n$ are the given quantities, $n$ is the total number of quantities).

**Example.** Given the numbers 83, 87, 81, 90.

$$\text{a. m.} = \frac{83 + 87 + 81 + 90}{4} = 85\frac{1}{4}$$

The *geometric mean* is obtained by multiplying the given quantities and taking the $n$th root of the product (where $n$ is the index of the root and is equal to the number of quantities taken):

$$\text{g. m.} = \sqrt[n]{a_1 a_2 \ldots a_n}$$

($a_1$, $a_2$, ..., $a_n$ are the given quantities, $n$ is the total number of them).

**Example.** Given the numbers 40, 50, 82.

$$\text{g. m.} = \sqrt[3]{40 \cdot 50 \cdot 82} = \sqrt[3]{164{,}000} \approx 54.74$$

The geometric mean is always less than the arithmetic mean (average) except for the case when all numbers are equal. Then the arithmetic mean is equal to the geometric mean. When there are only small fractional differences between the numbers, the difference between the arithmetic mean and the geometric mean is small compared to the numbers.

Averaging (computing arithmetic means) is of great importance in all branches of practical work.

**Example 1.** The distance between two points is measured with a 10-metre tape measure having centimetre divisions. Ten measurements are taken, which, in metres, are: 62.36, 62.30, 62.32, 62.31, 62.36, 62.35, 62.33, 62.32, 62.38, 62.37. The diversity of results is due to accidental inaccuracies in the measuring process. These findings are then averaged:

$$\text{a. m.} = (62.36 + 62.30 + 62.32. + 62.31 + 62.36 + 62.35 + 62.33 + 62.32 + 62.38 + 62.37) : 10 = 62.34$$

This number is a more reliable value of the distance than the numbers obtained in the measurements because random

(accidental) errors are nearly always balanced when computing the average (see Sec. 61 below).

Example 2. Height measurements are taken of 1000 persons. These are averaged. The result is the "average height" of the persons involved. It does not, generally speaking, signify the actual height of a given person. But if the measurements are taken of a large number of other people, the average height will again be just about the same. Quite naturally, it may happen that either giants or pigmies predominate in a sample of 1000. However, out of all conceivable cases these exceptional ones constitute an insignificantly small percent. Hence, for all practical purposes we can take it that the average height of any group of 1000 persons will be almost the same. The arithmetic means obtained in mass measurements are termed *statistical means*. Statistical means are of considerable practical importance. For example, knowing the average milk yield of a cow of a definite breed under specific feeding conditions, etc., it is possible to compute the yield of a herd by multiplying the average yield by the number of cows in the herd.

### 60. Abridged Calculation of the Arithmetic Mean

The numbers involved in computing an arithmetic mean are ordinarily bunched rather close together. If such is the case, computing the average (arithmetic mean) can be greatly simplified by the following technique.

(1) Take an arbitrary number close to the given numbers. If the given numbers differ from one another in the last digit alone, it is preferable to make the last digit of the selected number zero; if the given numbers differ in the last two digits, it is convenient to take a number ending in two zeros, and so on.

(2) Subtract this number in succession from each of the given ones.*

(3) Take the arithmetic mean of the differences thus found.

(4) Add the mean to the chosen number.

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* Both positive and negative numbers can be expected (see Sec. 67 on negative numbers). To avoid this, take a number less than any of the given ones. However, the computations will be somewhat simpler if the chosen number is about midway between the extremes of the given numbers.

**Example.** Find the arithmetic mean of ten numbers: 62.36, 62.30, 62.32, 62.31, 62.36, 62.35, 62.33, 62.32, 62.38, 62.37 (cf. with the preceding example).

(1) Choose the number 62.30.

(2) Subtract 62.30 from the given numbers; we find the differences (in hundredths) to be 6, 0, 2, 1, 6, 5, 3, 2, 8, 7.

(3) Now take the average of these differences, which is 4 (hundredths).

(4) Add 0.04 to 62.30 to get 62.34. This is the desired arithmetic mean (average).

### 61. Accuracy of the Arithmetic Mean

If the arithmetic mean is obtained from a comparatively small sequence of measurements (say, 10, like in Example 1, Sec. 59), it might well be that the actual value is somewhat different from the computed average. Then it is important to know how great this deviation can be. We are not speaking of the theoretically conceivable deviation (which can be arbitrarily great) but of the practically possible deviation (cf. Example 2, Sec. 59). The magnitude of the latter depends on the magnitude of the so-called *root mean square deviation.*

The root mean square deviation is the square root of the arithmetic mean of the squares of the deviations from the mean. It is denoted generally by the Greek letter $\sigma$ (sigma):

$$\sigma = \sqrt{\frac{(a_1-a)^2+(a_2-a)^2+\ldots+(a_n-a)^2}{n}} \qquad (A)$$

where $a=(a_1+a_2+\ldots+a_n):n$ (here $a_1$, $a_2$, ..., $a_n$ are the given numbers, $n$ is the total number of them, $a$ is their arithmetic mean, and $\sigma$ is the root mean square deviation).

**Note.** In formula (A) any one of the differences may be replaced by its reciprocal; this enables one to dispense with negative numbers (see Sec. 67 on negative numbers). Namely, when one of the given numbers is less than the mean, we take it for the subtrahend, and the mean for the minuend.

**Example.** Compute the root mean square deviation for the numbers of the preceding section. There, we found the mean to be 62.34. The deviations from the numbers 62.36, 62.30, etc. and their arithmetic mean are (in hundredths): 2, 4, 2, 3, 2, 1, 1, 2, 4, 3. The squares of these deviations are 4, 16, 4, 9, 4, 1, 1, 4, 16, 9. The arithmetic mean of the squares of the deviations is

$$\frac{4+16+4+9+4+1+1+4+16+9}{10}=6.8$$

(hundredths). The square root of this number, $\sqrt{6.8} \approx 3$ (hundredths); $\sigma = 0.03$.

If the number of measurements is approximately equal to 10, then the true value of the quantity cannot deviate from the arithmetic mean by more than the root mean square deviation $\sigma$. To be more precise, deviations exceeding $\sigma$ are possible only in exceptional cases, the number of which comes to about half a percent of all possible cases. In the example just considered, the true value cannot, practically speaking, deviate from the number 62.34 by more than 0.03. It therefore lies between $62.34 - 0.03 = 62.31$ and $62.34 + 0.03 = 62.37$.

If the number of measurements is substantially greater than ten, then the maximum practically possible deviation of the true value from the arithmetic mean will be less than $\sigma$. Namely, the deviation will not exceed the value $\frac{3\sigma}{\sqrt{n}}$ (where $n$ is the number of measurements). Thus, when the number of measurements is roughly 1000, the only practically possible deviations are those that do not exceed $0.1\sigma$.

### 62. Ratio and Proportion

The quotient of two numbers is termed their *ratio*. The term "ratio" was once applied only to cases when it was required to express one quantity as a fraction of another (homogeneous with the first); say, one length as a fractional part of another, one area as a fraction of another area, and so forth. These problems are handled by division (see Sec. 38). This explains why the special term "ratio" appeared: it once had a different meaning from "division", which referred to the division of a denominate number by an abstract number. This distinction is no longer made; for instance, we speak of the ratio of nonhomogeneous quantities, say the weight of a solid to its volume, etc. When speaking of homogeneous quantities, we often use percentage.

**Example.** A library has 10,000 books, of which 8000 are in Russian. What is the ratio of Russian books to the total number? $8000 : 10{,}000 = 0.8$. The desired ratio is 0.8, or 80%.

The dividend is called the *antecedent* of the ratio, the divisor, the *consequent*. In our example, 8000 is the antecedent, and 10,000 the consequent.

Two equal ratios form a *proportion*. Thus, if one library has 10,000 books, of which 8000 are Russian and another

library has 12,000 books, of which 9600 are Russian, then the ratio of Russian books to the total number of books in both libraries is the same: 8000:10,000 = 0.8; 9600:12,000 = 0.8. What we have is a proportion, which we write as follows: 8000:10,000 = 9600:12,000. In words, we say that 8000 is to 10,000 as 9600 is to 12,000. 8000 and 12,000 are the *extremes of the proportion*, and 10,000 and 9600 are the *means, of the proportion*.

The product of the means equals the product of the extremes. In our example, 8000·12,000 = 96,000,000; 10,000·9600 = 96,000,000. One of the extremes is equal to the product of the means divided by the other extreme. In the same way, one of the means is equal to the product of the extremes divided by the other mean. If

$$a:b=c:d$$

then

$$a=\frac{bc}{d}, \quad b=\frac{ad}{c}$$

and so on. In our example

$$8000=\frac{10,000\cdot 9600}{12,000}$$

This property is always used to compute the missing term of a proportion when the three other terms are known.

**Example.** $12:x=6:5$ ($x$ is the missing term, the unknown). $x=\frac{12\cdot 5}{6}=10$.

For practical applications of proportions see Sec. 64.

A proportion in which the means are equal is termed a *continued proportion*; for example, 18:6 = 6:2. The mean term of a continued proportion is the *geometric mean* (see Sec. 59) *of the extreme terms*. In our example, $6=\sqrt{18\cdot 2}$.

### 63. Proportionality (Variation)

The values of two different quantities can be interdependent. Thus, the area of a square depends on the length of the side, and conversely, the length of the side of a square is dependent on the area of the square.

Two mutually dependent quantities are termed *proportional* if the ratio of their values remains constant.

**Example.** The weight of kerosene is proportional to its volume; 2 litres of kerosene weigh 1.6 kg, 5 litres weigh

4 kg, 7 litres weigh 5.6 kg. The ratio of the weight to the volume is $\frac{1.6}{2} = 0.8$, $\frac{4}{5} = 0.8$, $\frac{5.6}{7} = 0.8$, etc.

The constant ratio of proportional quantities is called the *constant of proportionality (or constant of variation, or proportionality factor)*. It shows how many units of one quantity there are for every unit of another quantity; in our example, the number of kilograms that 1 litre of kerosene weighs (the specific weight of kerosene).

If two quantities are proportional, then any pair of values of one quantity forms a proportion with a pair of the corresponding values of the other taken in the same order. In our example, $1.6:4 = 2:5$, $1.6:5.6 = 2:7$, etc. Accordingly, we can define proportionality as follows: two quantities that depend on each other so that any increase in one causes an increase (in the same ratio) in the other are called *proportional quantities.*

If the dependence of two quantities is such that one increases as the other decreases (in the same ratio) then we have *inversely proportional quantities.* For example, the running time of a train between two stations is inversely proportional to the speed of the train. At a speed of 50 km/h, a train covers the distance between Moscow and Leningrad in 13 hours; going at 65 km/h, it covers the distance in 10 hours; that is when the speed increases in the ratio $\frac{65}{50} = \frac{13}{10}$, the running time diminishes in the ratio $\frac{13}{10}$.

If two quantities are inversely proportional, then any pair of values of one quantity forms a proportion with a pair of the corresponding values of the other taken in the reverse order. In our example, $65:50 = 13:10$.

The product of the values of two inversely proportional quantities remains unchanged. In our example, $50 \cdot 13 = 650$; $65 \cdot 10 = 650$ (650 km is the distance between Moscow and Leningrad).

### 64. Uses of Proportions. Interpolation

The solution of many problems involves proportional quantities. Application of the rules given in Sec. 62 mechanizes the solution of such problems, reducing them to a unified procedure, as illustrated in the examples given below.

**Example 1.** Fuel consumption at a factory was at 1.8 tons per 24 hours and at a cost of 3000 roubles per year.

Efficiency proposals reduced daily consumption to 1.5 tons. What should the budget plan of the next year be with respect to fuel consumption at this factory?

An unsophisticated solution to this problem would be:

(1) find the annual fuel consumption prior to the efficiency proposals: $1.8 \cdot 365 = 657$ (tons);

(2) find the cost of one ton of fuel: $3000 : 657 = 4.57$ (roubles);

(3) find the annual cost of the fuel following the introduction of the efficiency suggestion:

$$4.57 \cdot 1.5 \cdot 365 = 2500 \text{ (roubles)}$$

A much faster and easier solution can be found by noting that the daily consumption of fuel and the annual cost are proportional quantities (this is evident from the fact that an increase in the daily consumption produces a proportionate increase in the yearly cost; see Sec. 63).

**Scheme of solution:**

$$\begin{array}{ll} 1.8 \text{ tons} & 3000 \text{ roubles}, \\ 1.5 \text{ tons} & x \text{ roubles}, \end{array}$$

$$x : 3000 = 1.5 : 1.8,$$

$$x = \frac{3000 \cdot 1.5}{1.8} = 2500 \text{ roubles}$$

Although proportional relationships are encountered very often, most of the relationships in practical situations do not obey the law of proportionality. It is therefore all the more important to note that even for such quantities the procedure of computation using proportions is still meaningful. For instance, if we consider the variations of nonproportional quantities within a certain rather narrow range, these changes will, for all practical purposes, be proportional.

To illustrate, take a square. A side is not proportional to the area; for instance, a side of 2 m is associated with an area of 4 $m^2$; a side of 2.01 m, with an area of $(2.01)^2 = 4.0401 \approx 4.040$ ($m^2$); a side of 2.02 m, with an area of $4.0804 \approx 4.080$ ($m^2$), and so on. We thus see that the ratio of the sides (for instance, 2.01:1) is not equal to the ratio of the corresponding areas (4.040:1). However, the ratio of the *changes in the side* within the range we confine ourselves to is, for all practical purposes, equal to the ratio of the *changes in the area.*

Indeed, when the side increases from 2 m to 2.01 m, the change is 0.01 m, when it increases from 2 to 2.02 m, the

change is 0.02 m. The ratio of the changes 0.02:0.01 is 2. The corresponding changes in the area will (to within the third decimal) be: in the first case, 0.040, in the second 0.080. The ratio of the changes, 0.080:0.040, is also 2. Thus, the change in length is proportional to the change in area if the quantities are taken to three places of decimals. If a fourth decimal is taken, a slight deviation from proportionality will be noted. To avoid any deviation even in the fourth decimal place, regard the change in a side over a still smaller range (say from 1 m to 1.002 m instead of from 1 to 1.02 m). In practical work, we always have regard only for a definite number of decimals (three, four, and rarely five). For this reason we can consider the changes in the side and area of a square as proportional quantities. The situation is the same in an overwhelming majority of other cases. This circumstance is utilized in reading between the lines, so to speak, when we have to do with tables covering a relatively small number of data. We are able to pick out values which lie between the tabulated values.

**Example 2.** Take a table of square roots (see pages 14-17). Suppose we wish to find $\sqrt{63.2}$. The table does not have the number 63.2, but only 63, 64, 65, etc.

| Radicand | Square Root | Change in Square Root |
|---|---|---|
| 63 | 7.937 | |
| 64 | 8.000 | 0.063 |
| 65 | 8.062 | 0.062 |

We calculate (see third column) the change, in the value of the root when the radicand varies by unity from 63 to 64 and from 64 to 65. We see that the difference in these changes occurs only in the third decimal place (by one unit). Actually this difference is still less; it occurs only in the fourth decimal place and rounding off to three decimals gave rise to it in the third decimal place.

Now if we take only three decimals, all our changes will be just about the same, that is to say, within the range between 63 and 65 the changes in square roots taken to three decimals are proportional to the changes in the radicands. We therefore find $\sqrt{63.2}$ using the following scheme:

| Change in Radicand | Change in Square Root |
|---|---|
| 1 | 0.062 |
| 0.2 | $x$ |

$$x : 0.062 = 0.2 : 1,$$

$$x = \frac{0.062 \cdot 0.2}{1} = 0.012$$

Now we find $\sqrt{63.2}$ by adding to $\sqrt{63} \approx 7.937$ the number 0.012. This yields

$$\sqrt{63.2} \approx 7.949$$

Check by extracting the root to three decimals and you will see that all the decimals of our result are correct.

The procedure we have just discussed is called *interpolation* (an inserting between). In mathematics, interpolation signifies any procedure by means of which it is possible, in a table with a given number of tabulated values, to find certain intermediate numerical values not directly given in the table. The elementary kind of interpolation which we discussed above is termed *linear interpolation*.

Interpolation is extensively used when dealing with tables of almost any kind.

# ALGEBRA

## 65. The Subject of Algebra

The subject of algebra involves the study of equations (Secs. 79-81) and a number of other problems that developed out of the theory of equations. At the present time, when mathematics has split up into a number of specialized areas, the field of algebra includes only equations of a special kind, the so-called *algebraic equations* (see Sec. 83).* On the origin of the name "algebra" see Sec. 66.

## 66. Historical Survey of the Development of Algebra

**Babylonia.** The roots of algebra go deep into antiquity. About 4000 years ago, Babylonian scholars were already solving quadratic equations (Sec. 93) and systems of two equations, one of which was of second degree (Sec. 97). These equations were used in solving a diversity of problems in land measurement, construction of buildings and in military affairs.

The literal designations which we use today in algebra were unknown to the Babylonians who formulated their equations rhetorically.

**Greece.** The first syncopated (abridged) notations for unknown quantities are encountered in the writings of the ancient Greek mathematician Diophantus (2nd to 3rd century). For the unknown, Diophantus used the word "arithmos" (number), the second power of the unknown was denoted by "dunamis" (the word had many meanings: power, property, degree**). For the third power, Diophantus used the term

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* Note that the usual school course of algebra includes areas that are only remotely related to the theory of equations. Such, for example, as progressions and logarithms, which belong more to arithmetic than to algebra. Their inclusion in the course of algebra is justified on pedagogical grounds.

** "Dunamis" was translated into Arabic as "mal" meaning property. In the 12th century mathematicians in Western Europe translated "mal" into Latin as "census", which has the same meaning.

"kubos" (cube), for the fourth power we find (translated into English) square-square, for the fifth, square-cube, for the sixth, cube-cube. He denoted these quantities by the first letters of the corresponding names (we give them in Latin letters): *ar, du, cu, ddu, dcu, ccu*. To distinguish the unknowns from known quantities, the latter were accompanied by the designation "mo" (monades for "units"). Addition was not indicated in any way, an abbreviation was used for subtraction, equality was shown by "is" ("isos" means equal).

Neither the Babylonians nor the Greeks considered negative numbers. An equation like 3 *ar* 6 *mo is* 2 *ar* 1 *mo* ($3x+6=2x+1$) Diophantus called "inappropriate". When Diophantus transposed terms from one side of the equation to the other, he said that an addend becomes a subtrahend, and a subtrahend becomes an addend.

**China.** Chinese scholars were solving first-degree equations and systems of them and also quadratic equations 2000 years prior to the Christian era. They were acquainted with negative numbers and irrational numbers. Since each symbol in Chinese writing stands for a concept, there could be no syncopations in Chinese algebra.

At later periods, Chinese mathematics was enriched with new attainments. At the end of the 13th century, the Chinese were fully acquainted with the law of formation of binomial coefficients which today goes by the name of "Pascal's triangle" (see Sec. 136). In Western Europe this law was discovered by Stifel, 250 years later.

**India.** Hindu scholars made extensive use of syncopated notation for unknown quantities and their powers. These notations were the initial letters of the corresponding names (an unknown was called "so-much"; the names of various colours—black, blue, yellow, etc.— were used for a second, third, etc. unknown). Hindu scholars made much use of irrational and negative numbers (Greek mathematicians knew how to find approximate values of roots but eschewed irrationalities in algebra). A new addition to the family of numbers was zero, which came in with the negative numbers. Formerly it had been used solely for the absence of a number, as a placeholder.

**Arab-language countries. Uzbekistan, Tajikistan.** The Hindu authors wrote on algebraic problems in their astronomical works. It was in Arabic writings—the international language of the Muslim world—that algebra emerged as an independent discipline. The founder of algebra as a special branch of learning was the Central Asian scholar Mohammad of Kho-

rezmi, more generally known as al-Khowarizmi (dweller of Khorezmi). His algebraic work, composed in the 9th century A. D., bears the name "the science of the reunion and opposition", or, more freely, "the science of transposition and cancellation". "Transposition" denoted the transfer of a subtrahend from one side of the equation to the other where it becomes an addend; the "opposition" (or cancellation) was the gathering of unknowns on one side of the equation and the unknowns on the other side. The Arabic for "transposition" is "al-jabr". Whence the name "algebra".

Al-Khowarizmi and those that followed him made extensive use of algebra in commercial and monetary computations. Neither he nor any of the other mathematicians who wrote in Arabic made any use of abbreviations.* Neither did they recognize negative numbers. From Hindu sources they knew about negative numbers but considered their use insufficiently justified. This was true, but whereas the Hindu scholars were able to confine themselves to a single complete quadratic equation, al-Khowarizmi and his successors had to distinguish three cases ($x^2+px=q$, $x^2+q=px$, $x^2=px+q$, where $p$ and $q$ are positive numbers).

Central Asian, Persian and Arabic mathematicians enriched algebra in a variety of ways. In higher-degree equations they knew how to find approximate values of the roots to a very high degree of accuracy. Thus, the celebrated Central Asian philosopher, astronomer and mathematician al-Biruni (973—1048), also of Khorezmi, reduced the problem of computing the side of a regular nonagon inscribed in a given circle to the cubic equation $x^3=1+3x$ and found (in sexagesimal fractions) the approximate value $x = 1.52'45''47'''13''''$ to within $\frac{1}{60^4}$, which in decimal fractions is correct to seven decimal places. (The sexagesimal fraction can be read as one unit, 52 sixtieths, 45 three thousand six hundredths, etc.). The scholar Omar Khayyam (1036—1123) of Naishapur, the famous classic of Iranian and Tajik poetry (known in the West for his celebrated Rubaiyat) made a systematic study of equations of the third degree. Neither he nor any other of the mathematicians of the Muslim world were able to find expressions for the roots of a cubic equation in terms of the coefficients. However, Omar

* Abbreviations were hardly necessary since Arabic writing is brief: vowels are not written, consonants and semi-consonants are simple letters and often merge into a single symbol.

Khayyam developed a method by which it is possible (geometrically) to find the number of real roots of a cubic equation (he himself was only interested in positive roots).

**Medieval Europe.** In the 12th century, the "Algebra" of al-Khowarizmi was translated into Latin and studied in Europe. It marked the beginning of the development of algebra in European countries, at first under the strong influence of the science of the East. Syncopated notation appeared for unknowns and new problems involved in trading were solved, but no essential advances were made until the first third of the 16th century when the Italians del Ferro and Tartaglia found rules for solving cubic equations of the type $x^3 = px + q$, $x^3 + px = q$, $x^3 + q = px$, and Cardano, in 1545, demonstrated that any cubic equation can be reduced to one of these three types. At the same time, Ferrari, a pupil of Cardano, solved a quartic (fourth-degree) equation.

The rules for solving such equations were so complicated that improvements had to be made in notation. These took place gradually during a whole century. At the end of the 16th century, the French mathematician Viète introduced literal symbols for unknowns and for known quantities as well (the unknowns were denoted by capital vowels, the known quantities by capital consonants). Syncopated notation was introduced for operations as well. Different authors used different kinds. By the middle of the 17th century, algebraic symbolism, thanks to the efforts of the French scholar Descartes (1596—1650), took on the general outlines that it has today.

**Negative numbers.** During the 13th to 16th centuries, negative numbers were considered by Europeans only in exceptional cases. After the discovery of the solution of the cubic equation, negative numbers gradually came to be accepted in algebra, although they were called "false" numbers. In 1629 Girard (France) gave a geometric depiction of negative numbers that we still use today. About twenty years later, negative numbers were accepted generally.

**Complex numbers.** The introduction of complex numbers (Secs. 92, 98) was also connected with the discovery of the solution to the cubic equation.

Even before this discovery, in the solution of the quadratic equation $x^2 + q = px$ one encountered a case when it was required to find the square root of $\left(\frac{p}{2}\right)^2 - q$, where the quantity $\left(\frac{p}{2}\right)^2$ was less than $q$. It was concluded in

that case that the equation does not have any solutions. Of course, it is hard to imagine that the new (complex) numbers would be introduced at a time when even negative numbers were considered "false". Yet when solving a cubic equation by the Tartaglia rule it turned out that without operations involving imaginary numbers it is impossible to obtain a real root.

Let us go into this in more detail. By Tartaglia's rule a root of the equation

$$x^3 = px + q \tag{1}$$

is given by the expression

$$x = \sqrt[3]{u} + \sqrt[3]{v} \tag{2}$$

where $u$ and $v$ are solutions of the system

$$u + v = q, \quad uv = \left(\frac{p}{3}\right)^3 \tag{3}$$

For example, for the equation $x^3 = 9x + 28$ ($p = 9$, $q = 28$) we have

$$u + v = 28, \quad uv = 27$$

whence either $u = 27$, $v = 1$ or $u = 1$, $v = 27$. In both cases

$$x = \sqrt[3]{27} + \sqrt[3]{1} = 4$$

This equation does not have any other real roots.

But, as Cardano had noted, the system (3) may not have any real solutions, whereas equation (1) has a real root and, what is more, a positive root. Thus, the equation $x^3 = 15x + 4$ has the root $x = 4$, but the system

$$u + v = 4, \quad uv = 125$$

has the complex roots: $u = 2 + 11i$, $v = 2 - 11i$ (or $u = 2 - 11i$, $v = 2 + 11i$)..

Bombelli (1572) was the first to shed light on this mysterious phenomenon. He pointed out that $2 + 11i$ is the cube of $2 + i$, and $2 - 11i$ is the cube of $2 - i$; hence we can write $\sqrt[3]{2 + 11i} = 2 + i$, $\sqrt[3]{2 - 11i} = 2 - i$ and then formula (2) yields $x = (2 + i) + (2 - i) = 4$.

It was now impossible to ignore complex numbers. However, the theory of complex numbers developed slowly. As late as the 18th century, famous mathematicians argued about how to find the logarithms of complex numbers.

Although complex numbers helped to obtain a wide range of important facts involving real numbers, their very existence seemed doubtful. Exhaustive rules for operating with complex numbers were given in the middle of the 18th century by Euler, one of the greatest mathematicians in the history of science. At the turn of the 19th century, Wessel of Denmark and Argand of France gave a geometrical representation of complex numbers (Sec. 104; the first steps in this direction were taken by Wallis of England in 1685). But the work of Wessel and Argand was disregarded and only in 1831, when this method was developed by the great German mathematician Gauss, was it accepted generally.

After solutions had been found for equations of the third and fourth degree, mathematicians strenuously sought the formula for solving the quintic (fifth-degree) equation. However, Ruffini (Italy) proved, at the turn of the 19th century, that the literal fifth-degree equation $x^5+ax^4+bx^3+cx^2+dx+e=0$ cannot be solved algebraically; more precisely, it is impossible to express any root of it in terms of the literal quantities $a$, $b$, $c$, $d$, $e$ using the six algebraic operations of addition, subtraction, multiplication, division, involution and evolution (Ruffini's proof was not without fault, and in 1824 Abel of Norway gave a flawless proof).

In 1830 Galois (France) demonstrated that no general equation whose degree exceeds 4 can be solved algebraically.

Nevertheless, every $n$th-degree equation has (if we consider complex numbers as well) $n$ roots, some of which may be equal. This was known to mathematicians as early as the 17th century (it stemmed from the analysis of numerous particular cases), but only at the end of the 18th century was the theorem mentioned above proved by Gauss.

The problems that engaged algebraists in the 19th and 20th centuries for the most part go beyond the range of elementary mathematics. Suffice it to note that in the 19th century many methods were developed for approximate solution of equations. In this direction, important results were obtained by the great Russian mathematician N. I. Lobachevsky.

### 67. Negative Numbers

The first numbers known to man were the natural numbers (Sec. 16). But these numbers do not suffice even in the simplest cases. Indeed, in the general case, one natural num-

ber cannot be divided by another if we confine ourselves to natural numbers alone. Yet situations arise in which we have to divide, say, 3 by 4, 5 by 12, and the like. Without the introduction of fractions, the division of natural numbers is impossible; fractions make this operation possible.

But the operation of subtraction still remains impossible in certain cases even after the introduction of fractions: we cannot subtract a larger number from a smaller number, say 5 from 3. In everyday life we do not need to perform such subtraction and so for a very long time that operation was considered not only impossible but even senseless.

The development of algebra demonstrated that such an operation is necessary (see Sec. 68 below) and it was put to use by scholars of India in about the 7th century, by Chinese scholars earlier still. Hindu scholars, seeking to find instances of such subtraction, came to an interpretation of it from the point of view of trade transactions. If a merchant has 5000 roubles and buys 3000 roubles worth of goods, he has $5000-3000=2000$ roubles. But if he has 3000 and buys 5000 roubles worth of goods, then he is 2000 roubles in debt. It was considered, accordingly that we subtract 5000 from 3000, the result being a dotted 2000, which meant "two thousand in debt".

This interpretation was artificial because a merchant never found the sum of his debt by subtracting 3000—5000, he always performed the subtraction thus: 5000—3000. What is more, this could serve—with a stretch of the imagination—only to explain the rules of addition and subtraction of "dotted numbers", but it could not account for the rules of multiplication and division (see Sec. 69 on rules of operations). Still and all, this interpretation remained for a long time in textbooks and manuals, and even today it occasionally appears.

The "impossibility" of subtracting a number from a smaller one is due to the fact that the natural numbers (positive whole numbers) form an infinite sequence in only one direction. If we subtract successively 1 from, say, the number 7, we get

$$6,\ 5,\ 4,\ 3,\ 2,\ 1$$

One more subtraction yields an absence of any number, and from there on there is nothing to subtract from. If we want to make subtraction possible in all cases, we must: (1) consider the "absence of a number" as a number (zero); (2) consider it possible to subtract another unit from this number, etc.

We thus generate new numbers which today look like this:

$$-1,\ -2,\ -3,\ \text{etc.}$$

These numbers are termed *negative integers (or negative whole numbers)*. The minus sign reminds us of the origin of a negative number from the successive subtraction of unity. This sign is called a sign of quantity to distinguish it from the subtraction sign, which is called an operational sign.

The introduction of negative integers implies the need for negative fractional numbers. If we take it that $0-5=-5$, then we must also accept the fact that $0-\frac{12}{7}=-\frac{12}{7}$. The number $-\frac{12}{7}$ is a negative fractional number.

In contradistinction to negative numbers (integral and fractional), the numbers (integral and fractional) which are studied in arithmetic are called *positive numbers*. To bring out this distinction still more, we can affix a plus sign to any positive number, in which case this is a sign of quantity (and not the sign of an operation). For example, the number 2 can be written as $+2$. *

Together, negative and positive numbers are called signed numbers in school textbooks. The generic term in scientific terminology for these numbers, together with zero, is *rational numbers*. The meaning of this term will become clear when we discuss the concept of an irrational number (see Sec. 91). Just as, prior to the introduction of negative numbers, there were no positive numbers and the number $\frac{3}{4}$ was simply a fraction and not a positive fraction, so prior to the introduction of irrational numbers, the numbers $+5$, $-5$, $-\frac{3}{4}$, $+\frac{3}{4}$, etc., were simply positive and negative numbers and fractions, and not rational numbers.

### 68. Negative Numbers (History and Rules of Operation)

For the student, probably the most difficult item in algebra is that devoted to operations involving negative numbers. This is not because the rules are complicated.

* The fact that the signs of operations and the signs of quantity are the same (+ and −) is an advantage with respect to computations, but the beginner finds this rather complicated. It is therefore advisable, at the beginning, to distinguish between the operational sign and the sign of quantity, and write $\overline{2}$ for a negative two instead of $-2$. This is done in logarithmic computations (see Sec. 129 et seq.).

Quite the contrary, they are simple. The difficulty is twofold: (1) Why introduce negative numbers? (2) Why are the operations involving them what they are and not something else? For instance, it is often hard to grasp why multiplication and division of a negative number by a negative number yields a positive number.

These questions usually arise because negative numbers are usually introduced before equations, and the rules for operating with negative numbers are not re-examined. Actually, it is in connection with the solution of equations that both questions can be answered. Historically speaking, that is exactly how the negative numbers arose. If there had been no equations, there would not have been any need for negative numbers.

For a long time equations were studied without the aid of negative numbers. This was extremely inconvenient. It was to overcome these inconveniences that negative numbers were introduced. It is worth noting that for a long time many outstanding mathematicians refused to use them and only grudgingly gave way to the inevitable. Even Descartes was still calling them "false numbers".

A simple example will serve to illustrate the nature of these inconveniences. When solving an equation of the first degree in one unknown, say,

$$7x-5=10x-11$$

we transpose the terms so that the knowns are on one side and the unknowns are on the other. In such an operation, the signs are reversed. Collecting unknown quantities in the right member of the equation and known quantities in the left member, we get

$$11-5=10x-7x, \quad 6=3x, \quad x=2$$

These manipulations can be carried out without invoking any negative numbers at all and considering the + and − signs as signs of addition and subtraction, and not as the signs of positive and negative numbers. But then we have to think over the question of which side to move the unknown terms, because if, say, in the above equation the unknown terms are transposed to the left side, we get

$$7x-10x=5-11$$

Without negative numbers, we cannot subtract 11 from 5, neither can we subtract $10x$ from $7x$, which means we cannot get ahead with the solution. Now it is not always so easy to see, beforehand, how to avoid this situation, especially if there are a large number of terms. A computor has to be ready to do a double amount of work in transposing terms to the proper side. It was to make the computational process more efficient that negative numbers were introduced. Indeed, if we agree to consider "possible" the "impossible" subtraction $5-11$, and denote the result by $-6$, and make the subtraction $7x-10x$ yield $-3x$, then we obtain

$$-3x=-6$$

Whence $x=-6:-3$.

Now it turns out that when introducing negative numbers, we have to set up the rule that in the division of a negative number ($-6$) by a negative number ($-3$), the quotient is a positive number (2). This is so because the quotient must yield the value of the unknown quantity $x$, which was found earlier in a different manner (without using negative numbers) and proved to be equal to 2.

That, in rough outline, was how negative numbers were first introduced: the aim was to rationalize the process of computation. The rules involving negative numbers emerged from the introduction of this more efficient technique into computational procedures.

Numberless tests and years of using negative numbers have demonstrated the extreme effectiveness of this technique which has found brilliant applications in all spheres of science and engineering. Everywhere, the introduction of negative numbers permits embracing, in a single rule, phenomena that would require dozens of rules if we confined ourselves to positive numbers.

To summarize, the two questions posed above may be answered as follows: (1) negative numbers were introduced so as to dispose of certain difficulties arising principally in the solution of equations; (2) the rules involving them follow from the necessity to coordinate the results obtained by means of negative numbers with those obtained without them.

All these rules (see Sec. 69) can be established when considering the most elementary kinds of equations, in the same way that we set up a rule for the division of negative numbers.

### 69. Operations with Negative and Positive Numbers

The absolute value of a negative number is the positive number obtained simply by reversing the sign. The absolute value of $-5$ is $+5$, that is, 5. The absolute value of a positive number (zero included) is the number itself.

The absolute-value sign consists of two vertical bars that enclose the number whose absolute value is being taken. For example, $|-5|=5$, $|+5|=5$, $|0|=0$.

1. **Addition.** (a) To add two numbers with like signs, combine their absolute values and prefix the common sign.

**Examples.** $(+8)+(+11)=19$, $(-7)+(-3)=-10$.

(b) To add two numbers with unlike signs, find the difference between their absolute values and prefix the sign of the number whose absolute value is greater.

**Examples.** $(-3)+(+12)=9$, $(-3)+(+1)=-2$.

2. **Subtraction.** The subtraction of one number from another can be replaced by addition; in this case, the minuend retains its sign, and the sign of the subtrahend is reversed.

**Examples.**

$$(+7)-(+4)=(+7)+(-4)=3,$$
$$(+7)-(-4)=(+7)+(+4)=11,$$
$$(-7)-(-4)=(-7)+(+4)=-3,$$
$$(-4)-(-4)=(-4)+(+4)=0$$

**Note.** When performing addition and subtraction, especially when the operation involves several numbers, it is advisable to do as follows: (1) remove all brackets; to do this, affix a plus sign if the earlier sign in front of the brackets was the same as that inside the brackets, and a minus sign if it was opposite to that inside the brackets; (2) add the absolute values of all the numbers which now have the plus sign; (3) add the absolute values of all the numbers which now have the minus sign; (4) find their difference and affix the sign of the greater sum.

**Example.** $(-30)-(-17)+(-6)-(+12)+(+2)$;

(1) $(-30)-(-17)+(-6)-(+12)+(+2)=-30+17-6-12+2$,

(2) $17+2=19$,

(3) $30+6+12=48$,

(4) $48-19=29$

The result is the negative number $-29$ since the greater sum (48) was obtained by combining the absolute values of those numbers which had minus signs in the expression $-30+17-6-12+2$.

This expression may be regarded both as a sum of the numbers $-30$, $+17$, $-6$, $-12$, $+2$ and as the result of the following successive operations: the addition of 17 to $-30$, the subtraction of 6, the subtraction of 12 and, finally, the addition of 2. Generally, the expression $a-b+c-d$ and so on may be regarded as the sum of the numbers $(+a)$, $(-b)$, $(+c)$, $(-d)$ and also as the result of the following successive operations: the subtraction of $(+b)$ from $(+a)$, the addition of $(+c)$, the subtraction of $(+d)$, and so on.

**3. Multiplication.** To multiply two numbers, multiply their absolute values and to the product affix a plus sign if the signs of the factors are the same, and a minus sign if they are different.

**Scheme** (rule of signs in multiplication):

| | | | | |
|---|---|---|---|---|
| $+$ | $\cdot$ | $+$ | $=$ | $+$ |
| $+$ | $\cdot$ | $-$ | $=$ | $-$ |
| $-$ | $\cdot$ | $+$ | $=$ | $-$ |
| $-$ | $\cdot$ | $-$ | $=$ | $+$ |

**Examples.** $(+2.4)\cdot(-5)=-12$, $(-2.4)\cdot(-5)=12$, $(-8.2)\times(+2)=-16.4$.

When multiplying several factors together, the sign of the product is positive if the number of negative factors is even, and negative if the number of negative factors is odd.

**Example.**

$$\left(+\tfrac{1}{3}\right)\cdot(+2)\cdot(-6)\cdot(-7)\cdot\left(-\tfrac{1}{2}\right)=-14$$

(three negative factors)

$$\left(-\tfrac{1}{3}\right)\cdot(+2)\cdot(-3)\cdot(+7)\cdot\left(+\tfrac{1}{2}\right)=7$$

(two negative factors)

**4. Division.** To divide one number by another, divide the absolute value of the former by the absolute value of the latter and place a plus sign in front of the quotient if the signs of the dividend and divisor are the same, and

a minus sign if they are different (the scheme is the same as for multiplication).

**Examples.** $(-6):(+3)=-2$, $(+8):(-2)=-4$, $(-12):(-12)=+1$.

### 70. Operations with Monomials. Addition and Subtraction of Polynomials

A *monomial* is a product of two or more factors each of which is either a number, a letter or a power of a letter. For example, $2d$, $a^3b$, $3abc$, $-4x^2y^3$ are monomials. A single number or a single letter may also be regarded as a monomial.

Any one of the factors of a monomial may be taken as the coefficient of the monomial. The numerical factor (say, $-4$ in the expression $-4x^2yz^3$) is often taken as the coefficient. By isolating one of the factors as the coefficient, we simply wish to stress that the monomial is the result of multiplying the rest of the expression by the coefficient. By separating out the numerical factor as the coefficient, we wish to emphasize that the main role is played by the literal expression, which is repeated as an addend as many times as indicated by the numerical factor (coefficient).

Monomials are called similar if they are the same or if they differ solely in the coefficients. It is clear then that two monomials may be similar or dissimilar depending on what coefficients are taken. If the numerical factors are the coefficients, then like (or similar) monomials are those whose letter parts are the same. For example, the monomials $ax^2y^2$, $bx^2y^2$, $cx^2y^2$ are similar if the coefficients are taken to be $a$, $b$, $c$; the monomials $3x^2y^2$, $-5x^2y^2$, $6x^2y^2$ are similar if the coefficients are the numerical factors.

**Addition of monomials.** Generally speaking, the addition of two or more monomials can only be indicated; until we assign numbers to the letters, the sum of several terms cannot, as a rule, be simplified. The only simplification possible is in the case of similar terms; here, the sum of similar terms is a term whose coefficient is equal to the sum of their coefficients. This procedure is called collecting like terms.

**Example 1.** $3x^2y^2-5x^2y^2+6x^2y^2=4x^2y^2$.

**Example 2.** $ax^2y^2-bx^2y^2+cx^2y^2=(a-b+c)\,x^2y^2$.

**Example 3.** $4x^3y^2-3x^2y^2-2x^3y^2+6x^2y^2+5xy$
$=2x^3y^2+3x^2y^2+5xy$.

**Taking out a common factor.** The operation performed in Example 2 is called factoring by taking out a common factor. We say that $x^2y^2$ has been factored out. Essentially, taking out a common factor is the same as collecting like terms.

**Polynomials.** The sum of any number of monomials is called a *polynomial*. The addition of two or more polynomials is nothing other than the formation of a new polynomial that includes all the terms of all the original polynomials.

Subtracting one polynomial from another is the same as adding a polynomial whose terms are those of the original polynomial with signs reversed.

**Example.** $(4a^2+2b-2x^2y^2)-(12a^2-c)+(7b-2x^2y^2)$
$=4a^2+2b-2x^2y^2-12a^2+c+7b-2x^2y^2=-8a^2+9b$
$-4x^2y^2+c$

(like terms are underlined with the same number of lines).

**Multiplication of monomials.** Generally speaking, the multiplication of monomials can only be indicated (cf. the foregoing on the addition of monomials). A product of two or more monomials can be simplified only when they include numerical coefficients or powers of the same letters, in which case the exponents of the appropriate letters are combined and the numerical coefficients multiplied together.

**Example.** $5ax^2y^5(-3a^3x^4z)=-15a^4x^6y^5z$.

[we add the exponents of the letter $a$ $(1+3=4)$ and of the letter $x$ $(2+4=6)$].

**Division of monomials.** Generally speaking, the division of one monomial by another one can only be indicated. The quotient of two monomials can be simplified if the dividend and divisor contain numerical coefficients or certain powers of the same letters, in which case the exponent of the divisor is subtracted from the exponent of the dividend, and the numerical coefficient of the dividend is divided by the numerical coefficient of the divisor.

**Example.** $12x^3y^4z^5:4x^2yz^2=3xy^3z^3$ [the exponents of the letter $x$ are subtracted $(3-2=1)$, the same for $y$ $(4-1=3)$ and for $z$ $(5-2=3)$].

**Note 1.** If the exponents of some letter are the same in the dividend and in the divisor, the letter is dropped in the quotient because, divided by itself, it yields unity. Subtracting exponents we would get 0. We therefore agree to consider the zeroth power of any number to be the number 1.

**Example.** $4x^2y^3:2x^2y=2x^0y^2=2y^2$ $(x^0=1)$.

**Note 2.** If the exponent of some letter in the dividend is less than the exponent of the same letter in the divisor, subtraction yields a negative power of that letter. For details concerning negative powers, see Sec. 125. The result can also be given in the form of a fraction. Then the negative exponent is not present.

**Example.** $10x^2y^5 : 2x^6y^4 = 5x^{-4}y = \frac{5y}{x^4}$ because $x^{-4} = \frac{1}{x^4}$.

### 71. Multiplication of Sums and Polynomials

The product of a sum of two or more expressions by an expression is equal to the sum of the products of each of the terms by that expression:

$(a+b+c)\,x = ax+bx+cx$ (with brackets removed)

In place of the letters $a$, $b$, $c$ we could take any expressions, say, any monomials. The letter $x$ can also be replaced by any expression; if that expression is itself a sum of several terms, say $m+n$, then we have $(a+b+c)(m+n) = a(m+n)+b(m+n)+c(m+n) = am+an+bm+bn+cm+cn$. That is, the product of a sum by a sum is equal to the sum of all possible products of each term of one sum by each term of the other sum.

For example, this rule refers to the product of a polynomial by a polynomial:

$$(3x^2-2x+5)(4x+2) = 12x^3-8x^2+20x+6x^2-4x+10$$
$$= 12x^3-2x^2+16x+10$$

**Outline for multiplication:**

$$\begin{array}{r} 3x^2-2x+5 \\ 4x+2 \\ \hline 12x^3-8x^2+20x \\ 6x^2-4x+10 \\ \hline 12x^3-2x^2+16x+10 \end{array}$$

### 72. Formulas for Short-Cut Multiplication of Polynomials

The following special cases of multiplication of polynomials are frequently encountered and so should be memorized. It is particularly important to get into the habit of

using these formulas for cases when the letters $a$ and $b$ are replaced by more complicated expressions (say, monomials).

**1.** $(a+b)^2=a^2+2ab+b^2$. The square of the sum of two quantities is equal to the square of the first term, plus twice the product of the two terms, plus the square of the second term.

**Example 1.** $104^2=(100+4)^2=10{,}000+800+16=10{,}816$.

**Example 2.** $(2ma^2+0.1nb^2)^2=4m^2a^4+0.4mna^2b^2+0.01n^2b^4$.

Warning: $(a+b)^2$ is not equal to $a^2+b^2$.

**2.** $(a-b)^2=a^2-2ab+b^2$. The square of the difference between two quantities is equal to the square of the first term, minus twice the product of the two terms, plus the square of the second term. This formula may be regarded as a special case of the preceding one [in place of $b$ we take $(-b)$].

**Example 1.** $98^2=(100-2)^2=10{,}000-400+4=9604$.

**Example 2.** $(5x^3-2y^3)^2=25x^6-20x^3y^3+4y^6$.

Warning: $(a-b)^2$ is not equal to $a^2-b^2$ [see 3].

**3.** $(a+b)(a-b)=a^2-b^2$. The product of the sum and difference of two quantities is equal to the difference of their squares.

**Example 1.** $71\cdot69=(70+1)(70-1)=70^2-1=4899$.

**Example 2.** $(0.2a^2b+c^3)(0.2a^2b-c^3)=0.04a^4b^2-c^6$.

**4.** $(a+b)^3=a^3+3a^2b+3ab^2+b^3$. The cube of the sum of two quantities is equal to the cube of the first term, plus three times the product of the square of the first term by the second, plus three times the product of the first term by the square of the second, plus the cube of the second.

**Example 1.** $12^3=(10+2)^3=10^3+3\cdot10^2\cdot2+3\cdot10\cdot2^2+2^3=1728$.

**Example 2.** $(5ab^2+2a^3)^3=125a^3b^6+150a^5b^4+60a^7b^2+8a^9$.

Warning: $(a+b)^3$ is not equal to $a^3+b^3$ [see 6].

**5.** $(a-b)^3=a^3-3a^2b+3ab^2-b^3$. The cube of the difference of two quantities is equal to the cube of the first term, minus three times the product of the square of the first term by the second, plus three times the product of the first term by the square of the second, minus the cube of the second term.

**Example.** $99^3=(100-1)^3=1{,}000{,}000-3\cdot10{,}000\cdot1+3\times100\cdot1-1=970{,}299$.

Warning: $(a-b)^3$ is not equal to $a^3-b^3$ [see 7].

**6.** $(a+b)(a^2-ab+b^2)=a^3+b^3$. The product of the sum of two quantities by the "imperfect square of the difference" is equal to the sum of their cubes.

7. $(a-b)(a^2+ab+b^2)=a^3-b^3$. The product of the difference of two quantities by the "imperfect square of the sum" is equal to the difference of their cubes.

### 73. Division of Sums and Polynomials

The quotient obtained by dividing the sum of two or more expressions by an expression is equal to the sum of the quotients obtained by dividing each term by that expression:

$$\frac{a+b+c}{x}=\frac{a}{x}+\frac{b}{x}+\frac{c}{x}$$

$a$, $b$, $c$, $x$ are any expressions; if they are all monomials, that is, if division of a polynomial by a monomial is performed, then each of the quotients may be simplified (see Sec. 70).

**Example.** $\frac{3a^2b+11ab^2}{ab}=\frac{3a^2b}{ab}+\frac{11ab^2}{ab}=3a+11b.$

If $a$, $b$, $c$ are monomials and $x$ is a polynomial, that is if division of a polynomial by a polynomial is performed, the quotient cannot, generally speaking, be represented as a polynomial (just as the quotient obtained by dividing a whole number by a whole number cannot always be represented as a whole number). To put it differently, we cannot always find a polynomial which, when multiplied by the polynomial of the divisor, will yield the polynomial of the dividend.

**Example.** The quotient $\frac{a^2+x^2}{a+x}$ cannot be represented as a polynomial; the quotient $\frac{a^2-x^2}{a+x}$ can be represented as a polynomial: $\frac{a^2-x^2}{a+x}=a-x$.

In the general case, the division of a polynomial by a polynomial can be carried out so that a remainder is left over, just as in the case of the division of whole numbers. However, it is necessary to establish just what is meant by the division of polynomials involving a remainder. If we divide the positive integer 35 by the positive integer 4, we get 8 and a remainder of 3. The numbers 8 and 3 have the property that $4\cdot 8+3=35$, that is if $p$ is the dividend, $q$ the divisor, $m$ the quotient and $n$ the remainder, then $mq+n=p$. But this does not suffice to determine the quotient and remainder completely; thus, in our example ($p=35$, $q=4$) the numbers $m=6$, $n=11$, $m=4$, and $n=19$

have the same property. Also note that the number $n$ must be less than $q$. This cannot be literally carried over to the case of division of polynomials because the same expression may be greater for one set of values of the letters and smaller than the other for another set. A modification is required: in each of the polynomials, some one of the letters of its terms is taken as the principal letter. The highest power of this letter is called the *degree of the polynomial*. Then division in which a remainder occurs is defined as follows.

To divide a polynomial $P$ by a polynomial $Q$ means to find a polynomial $M$ (quotient) and a polynomial $N$ (remainder) that satisfy two requirements: (1) the equality $MQ+N=P$ must hold, (2) the degree of the polynomial $N$ must be lower than that of the polynomial $Q$.

**Note.** The remainder $N$ need not contain the principal letter at all; then we say that $N$ is of degree zero.

It is always possible to find polynomials $M$ and $N$ which satisfy these requirements uniquely for a given choice of the principal letter. However, they may differ if the choice of principal letter is different. The process of finding the quotient $M$ and the remainder $N$ is similar to the process of division (with a remainder) of one multidigit number by another. The role of higher- and lower-order digits is played by terms containing the principal letter to higher and lower powers. Before beginning the division process, the terms of the dividend and divisor are arranged in a descending order of powers of the principal letter.

**Outline of division:**

$$\begin{array}{lll} \text{(dividend)}\ 8a^3+16a^2-2a+4 & \big| & 4a^2-2a+1\ \text{(divisor)} \\ \underline{\mp 8a^3 \pm 4a^2 \mp 2a} & \big| & \overline{2a+5\ \text{(quotient)}} \\ 20a^2-4a+4 & & \\ \underline{-20a^2 \pm 10a \mp 5} & & \\ 6a-1\ \text{(remainder)} & & \end{array}$$

(1) Divide the first term of the dividend $8a^3$ by the first term of the divisor $4a^2$; the result $2a$ is the first term of the quotient.

(2) Multiply this term by the divisor $4a^2-2a+1$ to get $8a^3-4a^2+2a$ which is written under the dividend, similar terms under each other.

(3) Subtract the terms of the result from the corresponding terms of the dividend; bring down the next term of the dividend to get $20a^2-4a+4$.

(4) Divide the first term of the remainder $20a^2$ by the first term of the divisor to get 5, which is the second term of the quotient.

(5) Multiply the second term of the quotient by the divisor to get $20a^2-10a+5$, which is written under the first remainder.

(6) Subtract the terms of this result from the corresponding terms of the first remainder to get the second remainder $6a-1$. Its degree is less than the degree of the divisor, which means the division process is complete. We have a quotient of $2a+5$ and a remainder of $6a-1$.

**Alternation mode of division:**

$$
\begin{array}{lll}
 & 2x^2+3x^2-2x-4 & \text{(quotient)} \\
\text{(divisor) } 3x-2 & \overline{|\ 6x^4+5x^3-12x^2-8x+3} & \text{(dividend)} \\
 & \underline{6x^4-4x^3} & \\
 & +9x^3-12x^2 & \\
 & \underline{9x^3-6x^2} & \\
 & -6x^2-8x & \\
 & \underline{-6x^2+4x} & \\
 & -12x+3 & \\
 & \underline{-12x+8} & \\
 & -5 & \text{(remainder)}
\end{array}
$$

### 74. Division of a Polynomial by a First-Degree Binomial

If a polynomial containing $x$ is divided by a first-degree binomial $x-l$, where $l$ is some number (positive or negative), then we get a remainder which can only be a zero-degree polynomial (see Sec. 73), which is to say, some number $N$. The number $N$ may be sought without finding the quotient. Namely, this number is equal to that value of the dividend which the latter assumes when $x=l$.

**Example 1.** Find the remainder left after dividing the polynomial $x^3-3x^2+5x-1$ by $x-2$. Substituting $x=2$ into the given polynomial, we find $N=2^3-3\cdot 2^2+5\cdot 2-1=5$.

Indeed, performing the division, we find the quotient $M=x^2-x+3$ and a remainder $N=5$.

**Example 2.** Find the remainder obtained from dividing the polynomial $x^4+7$ by $x+2$. Here, $l=-2$. Substituting $x=-2$ into $x^4+7$, we find $N=(-2)^4+7=23$.

This property of a remainder is called the *remainder theorem*. It was discovered by the French mathematician

Bézout (1730—1783). The remainder theorem reads: when a polynomial

$$a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m$$

is divided by $x-l$, the remainder obtained is

$$N = a_0l^m + a_1l^{m-1} + a_2l^{m-2} + \ldots + a_m$$

**Proof.** By the definition of division (Sec. 73), we have

$$a_0x^m + a_1x^{m-1} + \ldots + a_m = (x-l)\,Q + N$$

where $Q$ is a polynomial and $N$ is a number. Substitute $x=l$; the term $(x-l)\,Q$ vanishes and we get

$$a_0l^m + a_1l^{m-1} + \ldots + a_m = N$$

**Note.** It may happen that $N=0$. Then $l$ is a root of the equation

$$a_0x^m + a_1x^{m-1} + \ldots + a_m = 0 \qquad (1)$$

**Example.** The polynomial $x^3+5x^2-18$ leaves no remainder when divided by $x+3$. The quotient is $x^2+2x-6$. Hence, $-3$ is a root of the equation $x^3+5x^2-18=0$. Indeed, $(-3)^3+5(-3)^2-18=0$.

Conversely, if $l$ is a root of equation (1), then the left member of this equation is exactly divisible by $x-l$.

**Example.** The number 2 is a root of the equation $x^3-3x-2=0$ ($2^3-3\cdot2-2=0$). Hence, the polynomial $x^3-3x-2$ is exactly divisible by $x-2$. Indeed,

$$(x^3-3x-2):(x-2) = x^2+2x+1$$

### 75. Divisibility of the Binomial $x^m \mp a^m$ by $x \mp a$

1. The difference between identical powers of two numbers is exactly divisible by the difference between the numbers, that is $x^m-a^m$ is divisible by $x-a$. This criterion, like the ones which follow, are a consequence of the remainder theorem (Sec. 74).

The quotient consists of $m$ terms and has the following form: $(x^m-a^m):(x-a) = x^{m-1}+ax^{m-2}+a^2x^{m-3}+\ldots+a^{m-1}$ (the exponents on $x$ steadily diminish by unity, whereas the exponents on $a$ increase by unity so that the sum of the exponents is constant and equal to $m-1$; all coefficients are equal to $+1$).

**Examples.**

$(x^2-a^2):(x-a)=x+a,$
$(x^3-a^3):(x-a)=x^2+ax+a^2,$
$(x^4-a^4):(x-a)=x^3+ax^2+a^2x+a^3,$
$(x^5-a^5):(x-a)=x^4+ax^3+a^2x^2+a^3x+a^4.$

2. The difference between identical even powers of two numbers is divisible not only by the difference between these numbers (Item 1) but also by their sum, i. e., $x^m-a^m$, for even $m$, is divisible both by $x-a$ and by $x+a$. In the latter instance, the quotient is of the form $x^{m-1}-ax^{m-2}+a^2x^{m-3}-\ldots$ (the plus and minus signs alternate).

**Examples.**

$(x^2-a^2):(x+a)=x-a,$
$(x^4-a^4):(x+a)=x^3-ax^2+a^2x-a^3,$
$(x^6-a^6):(x+a)=x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5.$

**Note.** Since the difference between even powers is divisible by $x-a$ and by $x+a$, it is also divisible by $x^2-a^2$.

**Examples.**

$(x^4-a^4):(x^2-a^2)=x^2+a^2,$
$(x^6-a^6):(x^2-a^2)=x^4+a^2x^2+a^4,$
$(x^8-a^8):(x^2-a^2)=x^6+a^2x^4+a^4x^2+a^6.$

The law of formation of quotients is obvious; it is readily subsumed under the law of Item 1, for example,

$$(x^8-a^8):(x^2-a^2)=[(x^2)^4-(a^2)^4]:(x^2-a^2)$$
$$=(x^2)^3+a^2(x^2)^2+(a^2)^2x^2+(a^2)^3$$

2a.. The difference between identical odd powers of two numbers is not divisible by the sum of the numbers.

For example, neither $x^3-a^3$ nor $x^5-a^5$ is divisible by $x+a$.

3. The sum of the same powers of two numbers is never divisible by the difference between these numbers.

For example, $x-a$ does not divide into $x^2+a^2$ or $x^3+a^3$ or $x^4+a^4$.

4. The sum of the same odd powers of two numbers is divisible by the sum of the numbers (in the quotient, the plus and minus signs alternate).

**Examples.**

$(x^3+a^3):(x+a)=x^2-ax+a^2,$
$(x^5+a^5):(x+a)=x^4-ax^3+a^2x^2-a^3x+a^4.$

4a. The sums of identical even powers of two numbers are not divisible by their difference (Item 3) or even by the sum of the numbers. For example, $x^2+a^2$ is not divisible either by $x-a$ or by $x+a$.

### 76. Factorization of Polynomials

A polynomial can sometimes be represented in the form of a product of two or more polynomials. This is by far not always possible but when it is possible, it is often very difficult to find the required factors. Its practical utility lies in the fact that it permits simplifying expressions (say, when common factors can be found in the numerator and denominator of a fraction; for examples, see the next section). The following are some elementary cases of the factorization of polynomials.

1. If all terms of a polynomial contain the same expression as a factor it can be taken outside the brackets (see Sec. 70, addition of monomials).

**Example 1.** $7a^2xy-14a^5x^3=7a^2x\,(y-2a^3x^2)$.

**Example 2.** $6x^2y^3-2uxy^2+4u^2xy=2xy\,(3xy^2-uy+2u^2)$.

2. It is sometimes possible to break the terms up into several groups, take out a factor in each group, and find the same expression in all sets of brackets. This expression can then be factored out; factorization of the polynomial is complete.

**Example 1.** $ax+bx+ay+by=x\,(a+b)+y\,(a+b)$
$=(a+b)\,(x+y)$.

**Example 2.** $\underline{10a^3}-\underline{\underline{6b^3}}+\underline{\underline{4ab^2}}-\underline{15a^2b}$

$=5a^2\,(2a-3b)+2b^2\,(2a-3b)=(2a-3b)\,(5a^2+2b^2)$.

**Note.** It is worth remembering that the expression $a-b$ can always be given in the form $-(b-a)$ so that what appears at first glance to be different expressions can easily be turned into the same expressions.

**Example 3.** $6ax-2bx+9by-27ay$

$=2x\,(3a-b)+9y\,(b-3a)=2x\,(3a-b)-9y\,(3a-b)$
$=(3a-b)\,(2x-9y)$.

3. The transformation explained in Item 2 can sometimes be carried out after first introducing new terms that mutually cancel or after breaking up one of the terms into two summands.

**Example 1.** $a^2-x^2=a^2+ax-ax-x^2$

$$=a(a+x)-x(a+x)=(a+x)(a-x)$$

(cf. formula 3 of Sec. 72).

**Example 2.** $p^2+pq-2q^2=p^2+2pq-pq-2q^2$

$$=p(p+2q)-q(p+2q)=(p+2q)(p-q).$$

4. The technique of Item 3 can sometimes be circumvented by using some of the factorization formulas obtained by manipulating the formulas of short-cut multiplication (Sec. 72), namely:

$$a^2+2ab+b^2=(a+b)^2;\quad a^2-2ab+b^2=(a-b)^2;$$
$$a^2-b^2=(a+b)(a-b),\text{ etc.}$$

**Example.** $4x^2+20xy+25y^2$. Using the first of these formulas ($a=2x$, $b=5y$), we get

$$4x^2+20xy+25y^2=(2x+5y)^2$$

Successful factorization of polynomials into the largest possible number of factors depends largely on one's dexterity in handling and combining the techniques enumerated above.

**Example.** $12+x^3-4x-3x^2=12-3x^2+x^3-4x$

$$=3(4-x^2)-x(4-x^2)=(4-x^2)(3-x)$$
$$=(2+x)(2-x)(3-x).$$

### 77. Algebraic Fractions

An *algebraic fraction* is an expression of the form $\frac{A}{B}$, where $A$ and $B$ denote any literal or numerical expressions, and the horizontal bar is the symbol for division. The dividend $A$ is called the numerator, the divisor $B$ is called the denominator. The fractions studied in arithmetic are a special case of algebraic fractions (in which the numerator and denominator are positive whole numbers). The rules for handling algebraic fractions are the same as those for the fractions of arithmetic (see Secs. 30-36). We therefore confine ourselves to a number of illustrative examples.

#### Reduction of Fractions

**Example 1.** The fraction $\frac{15a^3x^4}{21a^5x^3}$ may be reduced by dividing through by $3a^3x^3$; $\frac{15a^3x^4}{21a^5x^3}=\frac{5x}{7a^2}$.

**Example 2.** The fraction $\frac{2a^2-ab-3b^2}{2a^2-5ab+3b^2}$ may be reduced by

cancelling out $2a-3b$. To see this, factor the numerator and the denominator (see Sec. 76, Item 3):

$$\frac{2a^2-ab-3b^2}{2a^2-5ab+3b^2}=\frac{(2a-3b)(a+b)}{(2a-3b)(a-b)}=\frac{a+b}{a-b}$$

**Addition and Subtraction of Fractions**

**Example 1.** To add the fractions $\frac{m}{a^2b}+\frac{n}{ab^2}$ take $a^2b^2$ for the common denominator; the additional factors are $b$ for the first summand and $a$ for the second:

$$\frac{m}{a^2b}+\frac{n}{ab^2}=\frac{mb+na}{a^2b^2}$$

**Example 2.**

$$\frac{a-b}{2a^2-ab-3b^2}-\frac{a+b}{2a^2-5ab+3b^2}=\frac{a-b}{(2a-3b)(a+b)}-\frac{a+b}{(2a-3b)(a-b)}$$

$$=\frac{(a-b)^2-(a+b)^2}{(2a-3b)(a+b)(a-b)}$$

$$=\frac{-4ab}{(2a-3b)(a^2-b^2)}.$$

**Note.** Only in exceptional cases do multi-termed denominators have common factors, and if there are common factors, it usually requires a good deal of time to find them. Searching for such factors is a good exercise in developing algebraic habits and so the continued attention devoted to this work in textbooks is quite justified. However, their practical utility is slight and it is very often much better simply to take the product of the given denominators as the common denominator and not search for the simplest one.

**Multiplication and Division of Fractions**

**Example 1.** $\frac{4a^2b}{3c^2d}\cdot\frac{2c^2d^2}{ab^2}=\frac{8acd}{3b^2}$. Simplification can be carried out either prior to multiplication or afterwards.

**Example 2.** $\frac{x^2-a^2}{x^2-bx+cx-bc}:\frac{x^2-ax-cx+ac}{x^2-b^2}$

$$=\frac{(x^2-a^2)(x^2-b^2)}{(x-b)(x+c)(x-a)(x-c)}=\frac{(x+a)(x+b)}{(x+c)(x-c)}=\frac{(x+a)(x+b)}{x^2-c^2}.$$

**78. Proportions**

The definitions of a ratio and a proportion are given in Sec. 62. From the proportion $\frac{a}{b}=\frac{c}{d}$ follows $ad=bc$ (the

product of the means is equal to the product of the extremes); conversely, from $ad - bc$ follow the proportions

$$\frac{a}{b}=\frac{c}{d}, \quad \frac{a}{c}=\frac{b}{d}, \quad \frac{d}{b}=\frac{c}{a}$$

and others. All these proportions may be obtained from the original one, $\frac{a}{b}=\frac{c}{d}$, using the following rules.

1. In the proportion $\frac{a}{b}=\frac{c}{d}$, the means may be interchanged, the extremes may be interchanged, or both may be interchanged at the same time. We have

$$\frac{a}{c}=\frac{b}{d}, \quad \frac{d}{b}=\frac{c}{a}, \quad \frac{d}{c}=\frac{b}{a}$$

2. In a proportion the antecedent and the consequent of both ratios may be interchanged. From $\frac{a}{b}=\frac{c}{d}$ we get $\frac{b}{a}=\frac{d}{c}$. This proportion was obtained earlier (in the form of $\frac{d}{c}=\frac{b}{a}$). In the same way, nothing is altered by interchanging the antecedent and the consequent in the three proportions given above.

**Derived proportions.** If $\frac{a}{b}=\frac{c}{d}$, then the following proportions (called *derived proportions*) which are obtained from the given proportion hold true:

$$\frac{a+b}{a}=\frac{c+d}{c}, \quad \frac{a-b}{a}=\frac{c-d}{c}, \quad \frac{a+b}{b}=\frac{c+d}{d}, \quad \frac{a-b}{b}=\frac{c-d}{d},$$

$$\frac{a}{a+b}=\frac{c}{c+d}, \quad \frac{a}{a-b}=\frac{c}{c-d}, \quad \frac{b}{a+b}=\frac{d}{c+d}, \quad \frac{b}{a-b}=\frac{d}{c-d},$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}, \quad \frac{a+c}{b+d}=\frac{a}{b}=\frac{c}{d}, \quad \frac{a+b}{c+d}=\frac{a}{c}=\frac{b}{d},$$

$$\frac{a-b}{c-d}=\frac{a}{c}=\frac{b}{d}, \quad \frac{a-c}{b-d}=\frac{a}{b}=\frac{c}{d}$$

These and a multitude of other derived proportions can be combined into two basic forms:

$$\frac{ma+nb}{m_1a+n_1b}=\frac{mc+nd}{m_1c+n_1d}, \tag{1}$$

$$\frac{ma+nc}{m_1a+n_1c}=\frac{mb+nd}{m_1b+n_1d} \tag{2}$$

where $m$, $n$, $m_1$, $n_1$ are any numbers. Form (2) is obtained by the same rule as (1) if we first interchange the means in the given proportion.

Putting $m=n=m_1=1$, $n_1=0$ in formula (1), we get the derived proportion $\frac{a+b}{a}=\frac{c+d}{c}$; putting $m=n=m_1=1$, $n_1=0$ in (2), we have $\frac{a+c}{a}=\frac{b+d}{b}$ or, interchanging the mean terms, $\frac{a+c}{b+d}=\frac{a}{b}$, and so on.

### 79. Why We Need Equations

Computational problems are of two kinds: direct and indirect.

Here is an instance of a *direct problem:* what is the weight of a chunk of alloy which contains 0.6 $dm^3$ of copper (specific weight: 8.9 $kg/dm^3$) and 0.4 $dm^3$ of zinc (specific weight 7.0 $kg/dm^3$)? Solving, we find the weight of the copper ($8.9 \cdot 0.6 = 5.34$ (kg)), the weight of the zinc ($7.0 \cdot 0.4 = 2.8$ (kg)) and then the weight of the alloy ($5.34 + 2.8 = 8.14$ (kg)). These operations and their sequence are implied in the very statement of the problem.

Here is an instance of an *indirect problem:* an alloy consisting of copper and zinc of volume 1 $dm^3$ weighs 8.14 kg. Find the volumes of the copper and zinc in this alloy. Here, the statement of the problem does not indicate the operations that will lead to the solution. In what is called an arithmetic solution, considerable ingenuity is often required to find a plan of solution of an indirect problem. Each new problem requires setting up a fresh scheme. This is largely a waste of the computor's time. It was to rationalize the computing process that the method of equations, which is the basic subject of study in algebra, was created (see Sec. 65). The gist of the method is this.

1. The desired quantities are given special designations. For this purpose we use literal symbols (mostly the last letters of the Latin alphabet, $x$, $y$, $z$, $u$, $v$). Using these symbols and the signs of operations ($+$, $-$, etc.), we translate the conditions of the problem into mathematical language, that is, we express the relationships between the given quantities and the unknown quantities by mathematical symbols instead of words. Each such mathematical statement is an equation.

2. The next step is to solve the equation, that is, to find the values of the sought-for unknown quantities. The solution of an equation is a mechanical procedure, which is in strict accord with established rules. At this stage we no

longer have to take into account the specific features of the problem at hand, all we have to do is make use of firmly set rules and techniques. (One of the primary tasks of algebra is to derive these rules.)

Thus, equations are needed to mechanize the labour of the computor. After the equation has been set up, its solution can be obtained quite automatically (which is exactly what computing machines do at present). The difficulty of solving a problem lies in setting up the equation.

### 80. How to Set Up Equations

To form an equation is to express in mathematical form a relationship between the given (known) quantities of the problem and the sought-for (unknown) quantities. This relationship is sometimes so explicitly stated in the formulation of the problem that setting up the equation actually reduces to a word-for-word rehashing of the problem in the language of mathematical symbolism.

**Example 1.** Petrov received 16 roubles more for a job than half the sum which Ivanov got. Together their pay came to 112 roubles. How much did each get?

Denote by $x$ the share of Ivanov. Half his pay is $1/2x$; Petrov's pay for the month was $1/2x+16$; together they received a total of 112 roubles. In symbols we can write

$$(1/2x+16)+x=112$$

The equation has been set up. Solving it by the established rules (Sec. 84), we find that Ivanov got paid $x=64$ roubles; Petrov's pay was $1/2x+16=48$ roubles.

More often, however, it happens that the relationship between the known and unknown quantities is not explicitly stated in the problem and has to be built up out of the statement of the problem. Practical problems are almost all of that kind so that the case we gave above is of a rather artificial type hardly ever encountered in real situations.

It is therefore impossible to give exhaustive advice as to how equations are set up. However, as a starting piece of advice we offer the following. For the value of the unknown (or unknowns) take some number or numbers (at random) and make a check to see if the guess is a solution to the problem or not. If we can make the check and see if our guess is correct or incorrect (which is more likely),

then we can immediately set about forming the equation (or equations) we need. Namely, write down the very operations that were used to check the correctness of the randomly chosen number by introducing instead a literal symbol for the unknown. That is the equation we need.

**Example 2.** A piece of an alloy composed of copper and zinc weighs 8.14 kg and is 1 $dm^3$ in volume. How much copper is there in the alloy (specific weights: copper 8.9 $kg/dm^3$, zinc, 7.0 $kg/dm^3$)?

Take some number to express the desired volume of copper, say 0.3 $dm^3$. Now check to see how close a fit we get. Since 1 $dm^3$ of copper weighs 8.9 kg, 0.3 $dm^3$ will weigh $8.9 \cdot 0.3 = 2.67$ (kg). The volume of zinc in the alloy is $1 - 0.3 = 0.7$ ($dm^3$). This weighs $7.0 \cdot 0.7 = 4.9$ (kg). The total weight of zinc and copper comes out to $2.67 + 4.9 = 7.57$ (kg). But the weight of our piece is given as 8.14 kg. Our guess was wide of the mark, but at least we have an equation whose solution will yield the correct answer. In place of the guess (0.3 $dm^3$), denote the volume of copper (in $dm^3$) by $x$. In place of the product $8.9 \cdot 0.3 = 2.67$ take the product $8.9x$. This is the weight of the copper in the alloy. In place of $1 - 0.3 = 0.7$ take $1 - x$; this is the volume of the zinc. In place of $7.0 \cdot 0.7 = 4.9$, take $7.0(1 - x)$, which is the weight of the zinc. In place of $2.67 + 4.9$, take $8.9x + 7.0(1 - x)$, which is the total weight of the zinc and the copper, and which is given as 8.14 kg. Thus $8.9x + 7.0 \times (1 - x) = 8.14$. The solution of this equation (see Sec. 79) yields $x = 0.6$. The solution can be checked in a variety of ways, which yield a variety of equations, all of which however will result in the same solution. Such equations are called equivalent equations (see Sec. 82).

Quite naturally, after setting up equations has become a firmly established habit, there is no need to make these preliminary checks of conjectured numbers; simply start out by designating the unknown quantity by some letter ($x$, $y$, etc.) and then proceed as if this letter (the unknown) were the number we wish to verify.

### 81. Essential Facts about Equations

Two expressions, numerical or literal, joined together by an equality sign ($=$) form an equation (numerical or literal).

Every true numerical equation, and also every literal equation which holds true for all numerical values of the letters in them is termed an *identity*.

**Examples.** (1) The numerical equation $5 \cdot 3 + 1 = 20 - 4$ is an identity. (2) The literal equation $(a-b)(a+b) = a^2 - b^2$ is an identity since for all numerical values of $a$ and $b$ the right and left sides yield the same number. A literal equation is one in which all or some of the known quantities are expressed by letters, otherwise the equation is termed *numerical*.

It must be pointed out specially which letters of the equation are to be taken as known and which are to be regarded as unknown quantities. The unknowns are ordinarily denoted by $x$, $y$, $z$, $u$, $v$, $w$. According to the number of unknowns, equations are called equations in one, two, three, etc. unknowns.

To solve a numerical equation means to find numerical values of the indicated unknowns which turn the equation into an identity. These values are called the *roots* of the equation.

To solve a literal equation means to find expressions of the unknowns in terms of the known quantities given in the equation, which, when substituted for the unknowns, turn the equation into an identity. The expressions thus found are called the *roots of the equation*.

**Example 1.** $\frac{2}{3+x} = \frac{1}{2}x$ is a numerical equation in one unknown, $x$. For $x = 1$ the expressions $\frac{2}{3+x}$ and $\frac{1}{2}x$ form an identity that is, they yield one and the same number; $x = 1$ is the root of the equation.

**Example 2.** $ax + b = cx + d$ is a literal equation in one unknown, $x$; when $x = \frac{d-b}{a-c}$ it becomes an identity, since the expressions $a\frac{d-b}{a-c} + b$ and $c\frac{d-b}{a-c} + d$ yield the same numbers for all values of the letters $a$, $b$, $c$, $d$ $\left(\text{these expressions may be transformed to } \frac{ad-bc}{a-c}\right)$. The value $x = \frac{d-b}{a-c}$ is the root of the equation.

**Example 3.** $3x + 4y = 11$ is a numerical equation in two unknowns. When $x = 1$, $y = 2$ it becomes an identity: $3 \cdot 1 + 4 \cdot 2 = 11$. The values $x = 1$, $y = 2$ are roots of the equation; $x = -3$, $y = 5$ are also roots of the equation. The values $x = 2$, $y = 1\frac{1}{4}$ are also roots. This equation has infinitely many roots but it is not an identity because, for

example, when $x=2$ and $y=3$ the right and left members of the equation do not equal each other.

**Example 4.** $2x+3=2(x+1)$ is a numerical equation in one unknown. It does not become an identity for any values of $x$ whatever (the right member may be represented as $2x+2$: no matter what $2x$ equals, adding 2 to $2x$ cannot yield the same number as adding 3 to $2x$). This equation does not have any roots.

### 82. Equivalent Equations. Solving Equations

*Equivalent equations* are equations that have the same roots; for instance, $x^2=3x-2$ and $x^2+2=3x$ are equivalent since they both have the roots $x=1$ and $x=2$. The process of solving an equation consists mainly in replacing a given equation by another one that is equivalent to the first.

The basic techniques involved in solving equations are the following.

1. Replacement of an expression by an equivalent one. For example, the equation

$$(x+1)^2=2x+5$$

may be replaced by the equivalent equation

$$x^2+2x+1=2x+5$$

2. The terms of an equation may be transposed (carried over from one member, or side, of an equation to the other); in this process the sign of each term is reversed. For instance in the equation $x^2+2x+1=2x+5$ we can transpose all terms to the left member, the terms $+2x$ and $+5$ go from the right member to the left member with the sign reversed. The equation $x^2+2x+1-2x-5=0$ or, what is the same thing, $x^2-4=0$, is equivalent to the original equation.

3. Both members of an equation can be multiplied or divided by one and the same expression. It is important here to bear in mind that the new equation may not be equivalent to the preceding equation if the expression used to multiply or divide has the property of becoming equal to zero.

**Example.** Given the equation $(x-1)(x+2)=4(x-1)$. Divide both sides by $x-1$ to get $x+2=4$. This equation has a unique root, $x=2$, yet the original equation has the root $x=2$ and also the root $x=1$. This root was "lost" in the division by $x-1$. On the contrary, when we multiply

both members of the equation $x + 2 = 4$ by $(x-1)$ a new root, $x = 1$, appears in addition to the root $x = +2$. From this it does not in the least follow that we should not multiply or divide both members of an equation by an expression which is capable of becoming zero. The only thing we have to do, when performing such operations, is to note whether any old roots are lost or any new ones are introduced.

4. It is also possible to raise both members of an equation to the same power or extract the same root; however, the result may be equations that are not equivalent to the original one. For example, the equation $2x = 6$ has one root, $x = 3$; but the equation $(2x)^2 = 6^2$, or $4x^2 = 36$, has two roots: $x = 3$ and $x = -3$. Before transforming an equation, check to see whether some of the original roots are lost or new ones introduced. Most important is to be sure no old roots are lost. The introduction of new roots is not so dangerous because any root can be substituted into the original equation and verified on the spot to see whether it satisfies the equation or not.

### 83. Equations Classified

An equation is called *algebraic* if each of its members (sides) is a polynomial or monomial (single term) (see Sec. 70) with respect to the unknown quantities.

**Examples.** $bx + ay^2 = xy + 2^m$ is an algebraic equation in two unknowns, but the equation $bx + ay^2 = xy + 2^x$ is *not algebraic* because the right member is not a polynomial in the letters $x$ and $y$ (the term $2^x$ is not a monomial in the letter $x$).

*The degree of an algebraic equation.* Transpose all terms of an algebraic equation to one side and collect like terms; if the equation contains only one unknown, then the degree of the equation is the greatest exponent on the unknown. If the equation contains several unknowns, then for each term of the equation it is necessary to form the sum of the exponents of all unknowns. The largest sum is termed the *degree of the equation.*

**Example 1.** The equation $4x^3 + 2x^2 = 17x = 4x^3 - 8$ is a second-degree equation since we get $2x^2 - 17x + 8 = 0$ when all terms are transposed to the left side.

**Example 2.** The equation $a^4x + b^5 = c^5$ is an equation of the first degree since the highest degree of the unknown, $x$, is one.

**Example 3.** The equation $a^2x^5 + bx^3y^3 - a^8xy^4 - 2 = 0$ is an equation of the sixth degree since the sum of the exponents on the unknowns $x$ and $y$ comes to 5 for the first and third terms, 6 for the second and zero for the fourth; the largest sum is 6.

Equations whose solution reduces to the solution of an algebraic equation are sometimes also classed as algebraic. The degree of such an equation is the *degree of the algebraic equation to which it reduces.*

**Example 4.** $\frac{x^2+1}{x-1} = 2x$ is an equation of the second degree although the second degree of the unknown is not evident. However, if it is replaced by an equivalent algebraic equation (by getting rid of the fraction), it becomes $2x^2 - 3x - 1 = 0$. A first-degree equation in any number of unknowns is also called a *linear equation.*

### 84. First-Degree Equation in One Unknown

An equation of the first degree in one unknown may be transformed to one of the form $ax = b$, where $a$ and $b$ are given numbers or literal expressions containing known quantities. The solution (root) is of the form $x = \frac{b}{a}$. Manipulative difficulties occur only in carrying out transformations.

**Example 1.** $\frac{3x-5}{2(x+2)} = \frac{3x-1}{2x+5} - \frac{1}{x+2}$.

(1) Reduce the right member to a common denominator:

$$\frac{3x-5}{2(x+2)} = \frac{(3x-1)(x+2)-(2x+5)}{(2x+5)(x+2)}$$

(2) Remove brackets in the numerator of the right member and collect terms: $\frac{3x-5}{2(x+2)} = \frac{6x^2+3x-7}{(2x+5)(x+2)}$

(3) Multiply both members by $2(2x+5)(x+2)$ so as to clear the equation of fractions. (We leave the question as to whether or not extraneous roots have been introduced till after the solution.)

$$(3x-5)(2x+5) = 2(3x^2+3x-7)$$

(4) Removing brackets, we get

$$6x^2 + 5x - 25 = 6x^2 + 6x - 14$$

(5) Transpose all unknowns to the left member, all known quantities to the right, and collect like terms. We finally get $-x=11$, the root of the equation is thus $x=-11$.

Substitute this value into the original equation and verify that this root is not an extraneous root.

**Example 2.** $\frac{x^2}{(x-a)(x-b)}+\frac{(x-a)^2}{x(x-b)}+\frac{(x-b)^2}{x(x-a)}=3.$

(1) Reduce the left member to a common denominator:

$$x(x-a)(x-b)$$

(The additional factors are: $x$ for the first fraction, $x-a$ for the second, and $x-b$ for the third).

$$\frac{x^3+(x-a)^3+(x-b)^3}{x(x-a)(x-b)}=3$$

(2) We get rid of the fraction by multiplying both sides by $x(x-a)(x-b)$:

$$x^3+(x-a)^3+(x-b)^3=3x(x-a)(x-b)$$

(3) Removing brackets, we have

$$x^3+x^3-3ax^2+3a^2x-a^3+x^3-3bx^2+3b^2x-b^3$$
$$=3x^3-3ax^2-3bx^2+3abx$$

(4) Transpose unknown terms to the left and known quantities to the right member. Collecting like terms, we finally have

$$3a^2x-3abx+3b^2x=a^3+b^3 \text{ or } 3(a^2-ab+b^2)x=a^3+b^3$$

(5) We now find the root of the equation to be

$$x=\frac{a^3+b^3}{3(a^2-ab+b^2)}$$

This expression may be simplified by cancelling out $a^2-ab+b^2$:

$$x=\frac{a+b}{3}$$

### 85. A System of Two First-Degree Equations In Two Unknowns

After performing manipulations like those considered in the preceding section, an equation of the first degree in two unknowns $x$ and $y$ becomes $ax+by=c$, where $a$, $b$, $c$ are given numbers or literal expressions.

Taken separately, such an equation has infinitely many roots. Assign arbitrary values to one of the unknowns (say $x$) and the value of $y$ can be found from the equation in one unknown by substituting the value of $x$ into our equation. For instance, in the equation $5x+3y=7$ we can put $x=2$; we then get $10+3y=7$, whence $y=-1$.

However, if the unknowns $x$ and $y$ are connected by two (not one) equations of the first degree, then they will have infinitely many solutions only in exceptional cases (see Sec. 87). Generally, a system of two first-degree equations in two unknowns has only one set of solutions (solution set). It may also happen (in exceptional cases again) that it does not have any solutions at all (see Sec. 87).

Solving a system of two first-degree equations in two unknowns may be reduced, in a variety of ways, to the solution of one equation of the first degree in one unknown. Two such procedures are discussed in the next section.

Problems that lead to a system of two equations in two unknowns can always be solved by means of one equation in one unknown; however, much attention must then be paid to computations which, when a system of equations is employed, are handled routinely in the very process of solving the system. The same goes for problems involving three or more unknowns. They may be solved with the aid of one or two unknown quantities. The larger the number of unknowns involved, the simpler (generally speaking) it is to set up each one of the equations, but the more difficult is the process of solution of the system. Therefore, practical considerations suggest introducing as few unknowns as possible but in such a way that setting up the equations does not become too involved.

**Example.** A piece of alloy made of copper and zinc weighs 8.14 kg and is 1 $dm^3$ in volume. How much copper and zinc is there in the alloy (copper has a specific weight of 8.9 $kg/dm^3$, zinc, 7.0 $kg/dm^3$)? Denoting by $x$ and $y$ the unknown volumes of copper and zinc, we have two equations:

$$x+y=1, \tag{1}$$

$$8.9x+7.0y=8.14 \tag{2}$$

The first states that the total volume of copper and zinc (in $dm^3$) is equal to 1; the second states that the total weight (in kg) is equal to 8.14 ($8.9x$ is the weight of the copper; $7.0y$, the weight of the zinc). Solving the system of equations (1)—(2) using the general rules of Sec. 86, we find

$x = 0.6$, $y = 0.4$. We solved this same problem in Sec. 80 (Example 2) using only one unknown, $x$. The suggestions given in Sec. 80 hold true as well for setting up a system of equations in two or more unknowns.

### 86. Solving a System of Two First-Degree Equations in Two Unknowns

(a) **Solving by the substitution method.** This method consists in the following: (1) using one equation, find an expression of one of the unknowns, say $x$, in terms of the known quantities and the other unknown, $y$; (2) substitute this expression into the second equation, which then contains only one unknown, $y$; (3) solve the equation and find the value of $y$; (4) substitute the value of $y$ into the expression of the unknown $x$ which was found at the beginning of the solution. This yields the value of $x$.

**Example.** Solve the system of equations

$$\begin{aligned} 8x - 3y &= 46, \\ 5x + 6y &= 13 \end{aligned}$$

(1) Use the first equation to find the expression of $x$ in terms of the given numbers and the unknown $y$:

$$x = \frac{46 + 3y}{8}$$

(2) Substitute this expression into the second equation:

$$5 \cdot \frac{46 + 3y}{8} + 6y = 13$$

(3) Solve the resulting equation:

$$\begin{aligned} 5(46 + 3y) + 48y &= 104, \\ 230 + 15y + 48y &= 104, \\ 15y + 48y &= 104 - 230, \\ 63y &= -126, \quad y = -2 \end{aligned}$$

(4) Substitute the value of $y$ ($y = -2$) into the expression $x = \frac{46 + 3y}{8}$. This yields $x = \frac{46 - 6}{8} = 5$.

(b) **Solving by addition or subtraction.** This method consists in the following: (1) both members of one equation are multiplied by some factor; both members of the second equation are multiplied by another factor. These factors are chosen so that the coefficients of the unknowns in both

equations have the same absolute value after the operation. (2) Add or subtract the equations depending on whether the signs of the equalized coefficients have the same or different signs; in this way one of the unknowns is eliminated. (3) Solve the resulting equation in one unknown. (4) The other unknown can be found in the same way, but it is ordinarily easier to substitute the value of the first unknown into any one of the given equations and solve the resulting equation in one unknown.

**Example.** Solve the system of equations

$$\begin{aligned} 8x-3y&=46,\\ 5x+6y&=13 \end{aligned}$$

(1) The easiest way is to equalize the absolute values of the coefficients of $y$; multiply both sides of the first equation by 2; and both sides of the second by 1 (which is to say, leave it unchanged):

$$\begin{array}{l|l|l} 8x-3y=46 & 2 & 16x-6y=92,\\ 5x+6y=13 & 1 & 5x+6y=13 \end{array}$$

(2) Add the two equations:

$$\begin{array}{rr} & 16x-6y= \ 92\\ + & 5x+6y= \ 13\\ \hline & 21x \quad\ =105 \end{array}$$

(3) Solve the resulting equation:

$$x=\frac{105}{21}=5$$

(4) Substitute the value $x=5$ into the first equation to get

$$40-3y=46, \quad -3y=46-40, \quad -3y=6$$

Whence

$$y=\frac{6}{-3}=-2$$

The method of addition or subtraction is preferable to other methods: (1) when the absolute values of the coefficients of one of the unknowns are equal in the given equations (then the first of the steps in the solution process is not needed); (2) when it is seen at once that the numerical coefficients of one of the unknowns can be equalized by means of small integral factors; (3) when the coefficients of the equations contain literal expressions.

**Example.** Solve the system

$$\begin{aligned}(a+c)\,x-(a-c)\,y&=2ab,\\(a+b)\,x-(a-b)\,y&=2ac\end{aligned}$$

(1) Equalize the coefficients of $x$; multiply both sides of the first equation by $(a+b)$, of the second by $(a+c)$ to get

$$\begin{aligned}(a+c)(a+b)\,x-(a+b)(a-c)\,y&=2ab\,(a+b),\\(a+c)(a+b)\,x-(a-b)(a+c)\,y&=2ac\,(a+c)\end{aligned}$$

(2) Subtract the second equation from the first:

$$[(a-b)(a+c)-(a+b)(a-c)]\,y=2ab\,(a+b)-2ac\,(a+c)$$

(3) Solve the equation obtained:

$$y=\frac{2ab\,(a+b)-2ac\,(a+c)}{(a-b)(a+c)-(a+b)(a-c)}$$

This expression can be simplified considerably but the manipulations are involved. In the numerator and denominator, remove brackets, collect terms, factorize and cancel common factors to simplify the fraction:

$$\begin{aligned}y&=\frac{2a\,(ab+b^2-ac-c^2)}{(a^2-ab+ac-bc)-(a^2+ab-ac-bc)}\\&=\frac{2a\,[(ab-ac)+(b^2-c^2)]}{-2ab+2ac}\\&=\frac{2a\,[(b-c)\,a+(b-c)(b+c)]}{-2a\,(b-c)}\\&=\frac{2a\,(b-c)(a+b+c)}{-2a\,(b-c)}=-(a+b+c)\end{aligned}$$

(4) To find $x$, equalize the coefficients of $y$ in the original equations by multiplying the first by $(a-b)$ and the second by $(a-c)$. Subtracting one equation from the other, we solve the equation in one unknown and find

$$x=\frac{2ab\,(a-b)-2ac\,(a-c)}{(a-b)(a+c)-(a+b)(a-c)}$$

Performing the manipulations as in Item 3, we get $x=b+c-a$. Substituting of the value of $y$ into one of the original equations would have required unwieldy computations; this happens very often when solving literal equations.

## 87. General Formulas for and Special Cases of Solving Systems of Two First-Degree Equations in Two Unknowns

The solution of a system of equations of the form

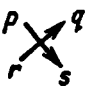
$$ax+by=c \tag{1}$$
$$a_1x+b_1y=c_1 \tag{2}$$

may be obtained much faster if use is made of certain general formulas that have been worked out. These formulas may be found by any method, for example, by the method of addition and subtraction. The solution has the following form:

$$x=\frac{b_1c-bc_1}{ab_1-a_1b}, \tag{3}$$

$$y=\frac{ac_1-a_1c}{ab_1-a_1b} \tag{4}$$

These formulas are easy to remember if we introduce for numerators and denominators the following symbolism. We agree to use the symbol $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$ to denote the expression $ps-rq$ which is obtained by cross multiplication:




and a subsequent subtraction of one product from the other (the product of the right-downwards diagonal has the plus sign). For example, the symbol $\begin{vmatrix} 5 & -8 \\ 2 & 1 \end{vmatrix}$ signifies $5\cdot1-2\times(-8)=5+16=21$.

The expression

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix}=ps-rq$$

is termed a *determinant of the second order* (in contrast to determinants of orders three, four, etc. which arise in the solution of systems of first-degree equations in three, four, etc. unknowns).

With the aid of this symbolism, formulas (3) and (4) look like this:

$$x=\frac{\begin{vmatrix} c & b \\ c_1 & b_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}} \quad (5), \qquad y=\frac{\begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}} \quad (6)$$

Each of the unknowns is equal to a fraction, the denominator of which is a determinant made up of the coefficients of the unknowns and the numerator is obtained from this determinant by replacing the coefficients of the corresponding unknown by the constant terms.

**Example.** Solve the system

$$\begin{aligned} 8x-3y&=46,\\ 5x+6y&=13 \end{aligned}$$

$$x=\frac{\begin{vmatrix} 46 & -3 \\ 13 & 6 \end{vmatrix}}{\begin{vmatrix} 8 & -3 \\ 5 & 6 \end{vmatrix}}=\frac{46\cdot 6+13\cdot 3}{8\cdot 6+5\cdot 3}=\frac{315}{63}=5,$$

$$y=\frac{\begin{vmatrix} 8 & 46 \\ 5 & 13 \end{vmatrix}}{\begin{vmatrix} 8 & -3 \\ 5 & 6 \end{vmatrix}}=\frac{8\cdot 13-5\cdot 46}{63}=\frac{-126}{63}=-2$$

An investigation shows that when solving the system (1)—(2), three essentially distinct cases arise.

(1) The coefficients of equations (1) and (2) are not proportional: $\frac{a}{a_1}\neq\frac{b}{b_1}$. Then, no matter what the constant terms, the equation has a unique solution given by formulas (3) and (4), or, what is the same, by formulas (5) and (6).

(2) The coefficients of (1) and (2) are proportional: $\frac{a}{a_1}=\frac{b}{b_1}$. Then it is important to know whether the constant terms as well are in that relation. If so, that is, if $\frac{a}{a_1}=\frac{b}{b_1}=\frac{c}{c_1}$, then the system of equations has an infinitude of solutions. The reason for this is that one of the equations is a consequence of the other, so that in actuality we are dealing with one equation and not two.

**Example.** In the system

$$\begin{aligned} 10x+6y&=18,\\ 5x+3y&=9 \end{aligned}$$

the coefficients of the unknowns $x$ and $y$ are proportional: $\frac{10}{5}=\frac{6}{3}=2$. The constant terms are in the same ratio $\frac{18}{9}=2$. Hence one of the equations is a consequence of the other; namely, the first is obtained from the second by multiplying

both members of the latter by 2. Any one of the infinity of solutions of one of the equations is a solution of the other.

(3) The coefficients of the equations are proportional: $\frac{a}{a_1}=\frac{b}{b_1}$ but the constant terms are not in that ratio. Then the system has no solution because the equations are contradictory.

**Example.** In the system

$$\begin{aligned} 10x+6y&=20,\\ 5x+3y&=9 \end{aligned}$$

the coefficients are proportional: $\frac{10}{5}=\frac{6}{3}=2$. The ratio of the constant terms is different from the ratio of the coefficients: $\frac{20}{9}=2\frac{2}{9}$. The system has no solution because if we multiply the second equation by 2, we get $10x+6y=18$ which contradicts the first equation, for one and the same expression $10x+6y$ cannot be equal to 18 and to 20.

### 88. A System of Three First-Degree Equations In Three Unknowns

If transformations similar to those indicated in Sec. 84 are carried out, a first-degree equation in three unknowns $x$, $y$, $z$ takes the form $ax+by+cz=d$, where $a$, $b$, $c$, $d$ are given numbers or literal expressions. One such equation, taken separately, or a system of two such equations has an infinity of solutions. In the general case, a system of three first-degree equations in three unknowns has one set of solutions. In exceptional cases (see below) it can have infinitely many solutions or none at all.

The solution of a system of three equations in three unknowns is based on the same techniques as used in the solution of a system of two equations in two unknowns, as will be seen from the following example.

**Example.** Solve the system of equations

$$\begin{aligned} 3x-2y+5z&=7, && (1)\\ 7x+4y-8z&=3, && (2)\\ 5x-3y-4z&=-12 && (3) \end{aligned}$$

Take two equations of this system, say (1) and (2), and assume that one of the unknowns (say $z$) has already been

found, that is, is known. Solving this system for $x$ and $y$ by the rules of Sec. 86, we get

$$x=\frac{17-2z}{13}, \quad y=\frac{59z-40}{26} \tag{4}$$

Substituting these expressions of $x$ and $y$ into (3), we get an equation in one unknown:

$$\frac{5(17-2z)}{13}-\frac{3(59z-40)}{26}-4z=-12$$

Solving this equation (see Sec. 84), we obtain $z=2$. Putting this value in (4), we find $x=1$, $y=3$.

The general formulas for solving the system

$$\left.\begin{aligned} ax+by+cz&=d\\ a_1x+b_1y+c_1z&=d_1\\ a_2x+b_2y+c_2z&=d_2 \end{aligned}\right\} \tag{5}$$

may be obtained by the same device. The solution will be complicated and hard to remember if written out in full, but it can be given a convenient and easily remembered form if we first introduce the concept of a *third-order determinant*.

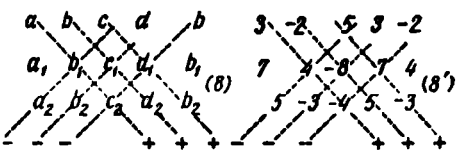
A determinant of third order, symbolized compactly as

$$\begin{vmatrix} a & b & c\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2 \end{vmatrix} \tag{6}$$

is simply the expression

$$ab_1c_2+bc_1a_2+ca_1b_2-cb_1a_2-ac_1b_2-ba_1c_2 \tag{7}$$

This expression need not be memorized since it is readily obtainable from its symbol (6) as follows: rewrite the array (6), adjoining on the right the first two columns. It then takes the form (8):




Draw diagonal lines [shown in (8) by the dashed lines] and write out the products of the letters on each of the six

diagonals. Affix the plus sign to the three products that represent the diagonals from upper left to lower right, and the minus sign to the other three products. Writing out these products horizontally, we get (7).

**Example 1.** Compute the third-order determinant

$$\begin{vmatrix} 3 & -2 & 5 \\ 7 & 4 & -8 \\ 5 & -3 & -4 \end{vmatrix} \tag{9}$$

Scheme (8) becomes scheme (8′).

The determinant (9) is then

$$3\cdot 4\,(-4) + (-2)\cdot(-8)\cdot 5 + 5\cdot 7\cdot(-3) - 5\cdot 4\cdot 5 - 3\cdot(-8)\cdot(-3) - (-2)\cdot 7\cdot(-4) = -48 + 80 - 105 - 100 - 72 - 56 = -301$$

Using determinants, we can represent the system (5) as

$$x = \frac{\begin{vmatrix} d & b & c \\ d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & d & c \\ a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}} \quad z = \frac{\begin{vmatrix} a & b & d \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}} \tag{10}$$

Here, each of the unknowns is equal to a fraction: the denominator is a determinant made up of the coefficients of the unknowns, and the numerator is obtained from this determinant by replacing the coefficients of the corresponding unknown by the constant terms.

**Example 2.** Solve the system of equations

$$\begin{aligned} 3x - 2y + 5z &= 7, \\ 7x + 4y - 8z &= 3, \\ 5x - 3y - 4z &= -12 \end{aligned}$$

The common denominator of the formulas (10) was computed in the example, it is $-301$. The numerator of the first of the formulas in (10) is obtained from (9) by replacing the first column by the column of constant terms. It looks like this:

$$\begin{vmatrix} 7 & -2 & 5 \\ 3 & 4 & -8 \\ -12 & -3 & -4 \end{vmatrix}$$

Computing it by (8) we get $-301$. Thus, we have $x=\frac{-301}{-301}=1$ (cf. the example on page 168). In the same way we find

$$y=\frac{-903}{-301}=3, \quad z=\frac{-602}{-301}=2$$

The system of equations (5) has a unique solution if the determinant made up of the coefficients of the unknowns is not equal to zero. Then formulas (10), with this determinant in the denominator, yield the solution of system (5). If the determinant made up of coefficients is zero, then formulas (10) become useless for the purpose of computation. In this case system (5) either has an infinity of solutions or none at all. There are an infinity of solutions if not only the determinant in the denominators but also the determinants in the numerators of formulas (10) vanish; it is important to note that if the determinant in the denominators and one of the determinants in the numerators are zero, then the other two determinants in the numerators must be zero. The existence of an infinity of solutions is due to the fact that one of the three equations (5) is a consequence of the other two [or even each of two of the equations (5) is a consequence of the third], so that actually we have not three but only two equations in three unknowns (or even one equation).

**Example 3.** In the system of equations

$$\left.\begin{aligned} 2x-5y+z&=-2\\ 4x+3y-6z&=1\\ 2x+21y-15z&=8 \end{aligned}\right\} \tag{11}$$

the determinant of coefficients is

$$\begin{vmatrix} 2 & -5 & 1\\ 4 & 3 & -6\\ 2 & 21 & -15 \end{vmatrix}=0$$

[see scheme (8)]. Taking one of the determinants in the numerators of formulas (10), say the determinant

$$\begin{vmatrix} -2 & -5 & 1\\ 1 & 3 & -6\\ 8 & 21 & -15 \end{vmatrix}$$

which is in the first one of (10) we find that it too is zero. The other two determinants in the second and third formulas of (10) need not be computed since they are certainly zero.

The system (11) has an infinite number of solutions; one of its equations (any one) is a consequence of the other two. For example, if we multiply the second equation by 2 and the first by —3 and combine the two equations, we get the third one.

The system (5) has no solutions at all if the determinant in the denominators of formulas (10) is zero but not one of the determinants in the numerators is zero. Here, it is sufficient to ascertain that one of the numerators is nonzero, then the other two will definitely be different from zero. The absence of solutions is due to the fact that one of the equations contradicts the other two (or even each one separately).

**Example 4.** Take the system of equations

$$\left.\begin{aligned} 2x-5y+z&=-2\\ 4x+3y-6z&=1\\ 2x+21y-15z&=3 \end{aligned}\right\} \qquad (12)$$

which differs from the system (11) solely in the value of the constant term in the last equation. The determinant of the coefficients therefore remains the same: equal to zero. However, the determinants in the numerators will be different. For instance, the numerator of the first of the formulas of (10) will be

$$\begin{vmatrix} -2 & -5 & 1\\ 1 & 3 & -6\\ 3 & 21 & -15 \end{vmatrix} = -135$$

It is not zero. The other two numerators are definitely not zero. The system (12) has no solutions. It is inconsistent because the first two equations yield $2x+21y-15z=8$ as a consequence (see Example 3); yet the third equation of (12) has the form $2x+21y-15z=3$. Thus one and the same expression is equal to both 3 and 8, which is impossible.

### 89. Laws of Exponents

1. The power of a product of two or more factors is equal to the product of the same power of each of the separate factors:

$$(abc\ldots)^n = a^n b^n c^n \ldots$$

**Example 1.** $(7\cdot 2\cdot 10)^2 = 7^2\cdot 2^2\cdot 10^2 = 49\cdot 4\cdot 100 = 19{,}600$.

**Example 2.** $(x^2-a^2)^3 = [(x+a)(x-a)]^3 = (x+a)^3(x-a)^3$ (cf. Sec. 72, Item 3).

Of more practical utility is the reverse transformation:

$$a^n b^n c^n \ldots = (abc\ldots)^n$$

where the product of the same powers of several quantities is equal to the same power of the product of the quantities.

**Example 3.** $4^3 \cdot \left(\frac{7}{4}\right)^3 \cdot \left(\frac{2}{7}\right)^3 = \left(4 \cdot \frac{7}{4} \cdot \frac{2}{7}\right)^3 = 2^3 = 8.$

**Example 4.** $(a+b)^2\,(a^2-ab+b^2)^2 = [(a+b)\,(a^2-ab+b^2)]^2 = (a^3+b^3)^2$

(cf. Sec. 72, Item 6).

2. The power of a quotient (or of a fraction) is equal to the quotient obtained by dividing the same power of the dividend by the same power of the divisor:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

**Example 5.** $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}.$

**Example 6.** $\left(\frac{a+b}{a-b}\right)^3 = \frac{(a+b)^3}{(a-b)^3}.$

The inverse transformation is: $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n.$

**Example 7.** $\frac{7.5^3}{2.5^3} = \left(\frac{7.5}{2.5}\right)^3 = 3^3 = 27.$

**Example 8.** $\frac{(a^2-b^2)^2}{(a+b)^2} = \left(\frac{a^2-b^2}{a+b}\right)^2 = (a-b)^2$

(cf. Sec. 72, Item 3).

3. When multiplying powers having the same base, add the exponents (cf. Sec. 70):

$$a^m a^n = a^{m+n}$$

**Example 9.** $2^2 \cdot 2^5 = 2^{2+5} = 2^7 = 128.$

**Example 10.** $(a-4c+x)^2\,(a-4c+x)^3 = (a-4c+x)^5.$

4. In dividing powers having the same base, subtract the exponent of the divisor from the exponent of the dividend (cf. Sec. 70):

$$\frac{a^m}{a^n} = a^{m-n}$$

**Example 11.** $12^5 . 12^3 = 12^{5-3} = 12^2 = 144.$

**Example 12.** $(x-y)^3 : (x-y)^2 = x-y.$

5. In finding a power of a power, multiply the exponents: $(a^m)^n = a^{mn}.$

**Example 13.** $(2^3)^2 = 2^6 = 64.$

**Example 14.** $\left(\frac{a^2b^3}{c}\right)^4 = \frac{(a^2)^4\,(b^3)^4}{c^4} = \frac{a^8 b^{12}}{c^4}.$

## 90. Operations Involving Radicals

In the formulas given below, the symbol $\sqrt{\ }$ denotes the absolute value of the root.

1. The value of a root remains unchanged if its index is increased $n$ times and the radicand is raised to the $n$th power:

$$\sqrt[m]{a} = \sqrt[m\cdot n]{a^n}$$

**Example 1.** $\sqrt[3]{8} = \sqrt[3\cdot 2]{8^2} = \sqrt[6]{64}$.

2. The value of the root remains unchanged if its index is reduced $n$ times and the $n$th root is taken of the radicand at the same time:

$$\sqrt[m]{a} = \sqrt[m:n]{\sqrt[n]{a}}$$

**Example 2.** $\sqrt[6]{8} = \sqrt[6:3]{\sqrt[3]{8}} = \sqrt{2}$.

**Note.** This property also holds true when the number $\frac{m}{n}$ is not integral; both properties are also valid when $n$ is fractional. But to see this we first have to extend the concept of a power and a root by introducing fractional exponents (see Sec. 125).

3. The root of a product of several factors is equal to the product of the roots (of the same index) of these factors:

$$\sqrt[m]{abc \ldots} = \sqrt[m]{a}\sqrt[m]{b}\sqrt[m]{c} \ldots$$

**Example 3.** $\sqrt[3]{a^6b^2} = \sqrt[3]{a^6}\sqrt[3]{b^2} = a^2\sqrt[3]{b^2}$.

The last transformation is based on Property 2.

**Example 4.** $\sqrt{48} = \sqrt{16\cdot 3} = \sqrt{16}\cdot\sqrt{3} = 4\cdot\sqrt{3}$.

Conversely, the product of roots of the same index is equal to the root (of that index) of the product of the radicands:

$$\sqrt[m]{a}\cdot\sqrt[m]{b}\cdot\sqrt[m]{c} \ldots = \sqrt[m]{abc \ldots}$$

**Example 5.** $\sqrt{a^3b}\cdot\sqrt{ab^3} = \sqrt{a^4b^4} = a^2b^2$.

4. The root of a quotient is equal to the quotient obtained by dividing the root of the dividend by the root of the

divisor (the indices of the roots are all assumed to be the same):

$$\sqrt[m]{a:b} = \sqrt[m]{a} : \sqrt[m]{b}$$

Conversely: $\sqrt[m]{a} : \sqrt[m]{b} = \sqrt[m]{a:b}$.

**Example 6.** $\sqrt[3]{27:4} = \sqrt[3]{27} : \sqrt[3]{4} = 3:\sqrt[3]{4}$.

5. To raise a root to a power, raise the radicand to that power:

$$\left(\sqrt[m]{a}\right)^n = \sqrt[m]{a^n}$$

Conversely, to extract the root of a power, raise the root of the base to that power:

$$\sqrt[m]{a^n} = \left(\sqrt[m]{a}\right)^n$$

**Example 7.** $\left(\sqrt[3]{a^2b}\right)^2 = \sqrt[3]{a^4b^2} = \sqrt[3]{a^3 \cdot ab^2} = a\sqrt[3]{ab^2}$.

**Example 8.** $\sqrt{27} = \sqrt{3^3} = (\sqrt{3})^3 = (\sqrt{3})^2 \cdot \sqrt{3} = 3\sqrt{3}$.

**6. Rationalizing the denominator or numerator of a fraction.** Computation of fractional expressions involving radicals is often simplified by what is termed "rationalizing" the numerator or denominator. This process signifies eliminating the radicals in the numerator or denominator.

**Example 9.** Let it be required to compute $\frac{1}{\sqrt{7}-\sqrt{6}}$ to 0.01. If the operations are carried out in the indicated order, we have (1) $\sqrt{7} \approx 2.646$, (2) $\sqrt{6} \approx 2.499$, (3) $2.646 - 2.449 = 0.197$, (4) $\frac{1}{0.197} \approx 5.10$. Four operations were required to obtain the answer; what is more, to obtain digits correct to hunredths required computing the roots to thousandths, otherwise we would have only two significant digits in the denominator of the fraction $\frac{1}{\sqrt{7}-\sqrt{6}}$ and so we could not obtain three correct significant digits (see Sec. 56).

But if we first multiply the numerator and denominator of the fraction by $\sqrt{7}+\sqrt{6}$, we get

$$\frac{1}{\sqrt{7}-\sqrt{6}} = \frac{\sqrt{7}+\sqrt{6}}{(\sqrt{7})^2-(\sqrt{6})^2} = \frac{\sqrt{7}+\sqrt{6}}{1}$$

The computation now requires only three operations, and the root need only be evaluated to hundredths:

(1) $\sqrt{7} \approx 2.65$, (2) $\sqrt{6} \approx 2.45$, (3) $\sqrt{7} + \sqrt{6} \approx 5.10$.

Here are some typical illustrations.

**Example 10.** $\frac{1}{\sqrt{5}} = \frac{\sqrt{7} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{\sqrt{35}}{5}$.

**Example 11.** $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a})^2-(\sqrt{b})^2} = \frac{a+2\sqrt{ab}+b}{a-b}$.

In these examples, we rationalized the denominator. In the two following examples we rationalize the numerator.

**Example 12.** $\frac{\sqrt{7}}{\sqrt{5}} = \frac{\sqrt{7} \cdot \sqrt{7}}{\sqrt{5} \cdot \sqrt{7}} = \frac{7}{\sqrt{35}}$.

**Example 13.** $\frac{\sqrt{35}-\sqrt{34}}{3} = \frac{\sqrt{35^2}-\sqrt{34^2}}{3(\sqrt{35}+\sqrt{34})} = \frac{1}{3(\sqrt{35}+\sqrt{34})}$.

The transformation in Example 12 is clearly unsuitable for computational purposes because evaluating the expression $\frac{7}{\sqrt{35}}$ requires division by a multidigit number, whereas the computation of $\frac{\sqrt{35}}{5}$ (see Example 10) requires division by a whole number. But the transformation in Example 13 is useful in that it permits computing the roots $\sqrt{35}$ and $\sqrt{34}$ to as many places as required in the answer, whereas in the original expression we would have had to extract the roots to a larger number of places (see Example 9). Thus, the general school practice of rationalizing the denominator every single time is a harmful formalistic tradition.

### 91. Irrational Numbers

The range of whole and fractional numbers is more than sufficient for purposes of mensuration (see Sec. 45). It is not, however, sufficient for the theory of mensuration.

For example, let it be required to determine *exactly* the length of the diagonal *AC* in the square *ABCD* (Fig. 1), the side of which is 1 metre long. The area of the square *ACEF* constructed on the diagonal is equal to twice the

area of $ABCD$ (the triangle $ACB$ is contained twice in $ABCD$ and four times in $ACEF$). Therefore, if $x$ is the desired length of $AC$, then $x^2=2$. However, no whole number and no fraction can satisfy this equation.

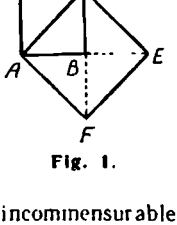


Fig. 1.

We can do one of two things: either give up attempting to express exactly lengths by numbers or introduce new numbers in addition to the whole and fractional numbers. After a long period of struggle, the latter point of view prevailed.

These new numbers, which represent the lengths of line segments that are incommensurable with the unit length (that is to say, line segments that cannot be expressed by any whole or fractional number) are called *irrational numbers*.* In contrast to irrational numbers, whole numbers (integers) and fractions were called *rational*. After the introduction of negative numbers (this was later; see Sec. 66), they too were split into rational and irrational.

Every rational number can be represented in the form $\frac{m}{n}$, where $m$ and $n$ are integers (positive or negative). Irrational numbers cannot be exactly represented in this form. However, every irrational number can be replaced approximately, to any degree of accuracy, by a rational number $\frac{m}{n}$; for instance, it is possible to find a decimal fraction (pure or mixed) which differs from a given irrational number by an arbitrarily small number.

The numbers $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{3+\sqrt{2}}$, $\sqrt{\sqrt[3]{5}+\sqrt{7}}$

and many other expressions involving rational numbers under the radical sign are irrational. These irrational numbers are said to be expressed in terms of radicals.

They do not however exhaust the range of irrational numbers. Up to the end of the 18th century, mathematicians were convinced that the root of any algebraic equation with

* The term "irrational" literally means "having no ratio". Originally, it did not refer to an irrational number but to quantities whose ratio is now expressed by an irrational number. Say, the ratio of the diagonal of a square to its side is now given by the number $\sqrt{2}$. Prior to the advent of irrational numbers, it was customary to say that the diagonals of a square cannot be related to its side.

rational coefficients could be expressed in terms of radicals if the root is not rational; later it was proved that this is true only for equations up to the fourth degree inclusive (see Sec. 66). As a rule, the irrational roots of equations of fifth and higher degree cannot be expressed in terms of radicals. Numbers which are the roots of algebraic equations with integral coefficients are called *algebraic numbers;* only in exceptional cases are algebraic numbers expressible by radicals, and still rarer are the cases when they are rational.

Algebraic numbers still do not exhaust the range of irrational numbers. Thus, for instance, the well-known number $\pi$ of geometry (see Sec. 152) is irrational, but it cannot be the root of any algebraic equation with integral coefficients. The same goes for the number $e$ (see Sec. 128) which is not algebraic. In other words, pi ($\pi$) and $e$ are not algebraic numbers.

An irrational number which cannot be the root of any algebraic equation with integral coefficients is called a *transcendental number.*

Up to 1929, only a few numbers had been proved to be transcendental; the transcendence of the number $e$ was proved in 1871 by the French mathematician Hermite. In 1882 the German mathematician Lindemann proved the transcendence of $\pi$. Academician A. A. Markov (1856—1922) proved the transcendence of $e$ and $\pi$ by a new method. In 1913 D.D. Mordukhai-Boltovskoi (1877—1952) pointed out a number of new transcendental numbers. It was still not known, however, whether such "ordinary" numbers as $3^{\sqrt{2}}$, $\sqrt{3}^{\sqrt{2}}$ were transcendental or not. The Soviet mathematicians A. O. Gelfond and R. O. Kuzmin (1891—1949) proved in 1929 and 1930 that all numbers of the form $\alpha^{\sqrt{n}}$, where $\alpha$ is an algebraic number not equal to zero or unity and $n$ is an integer, are transcendental. The numbers $3^{\sqrt{2}}$, $\sqrt{3}^{\sqrt{2}}$, etc., are precisely of this form. In 1934 Gelfond completed these studies. He proved the transcendence of all numbers of the form $\alpha^{\beta}$ where $\alpha$ and $\beta$ are arbitrary algebraic numbers (provided that $\alpha$ is neither 0 nor 1 and $\beta$ is irrational). For example, the number $\left(\sqrt[4]{5}\right)^{\sqrt[3]{2}}$ is transcendental.

From the transcendence of the numbers $\alpha^{\beta}$ there readily follows the transcendence of the decimal logarithms of all integers (except, of course, 1, 10, 100, 1000, etc.).

### 92. Quadratic Equations. Imaginary and Complex Numbers

An algebraic equation of the second degree is called a *quadratic* equation. The most general form of a quadratic equation in one unknown is

$$ax^2+bx+c=0$$

where $a$, $b$, $c$ are given numbers or literal expressions involving known quantities (note that the coefficient $a$ cannot be zero for otherwise the equation would not be quadratic, but only of degree one). Dividing both members of the equation by $a$, we get an equation of the form

$$x^2+px+q=0 \quad \left(p=\frac{b}{a},\ q=\frac{c}{a}\right)$$

A quadratic equation of this kind is called *reduced*; the equation $ax^2+bx+c=0$ (where $a \neq 1$) is called *unreduced*. If one of the quantities $b$, $c$ or both are zero, then the quadratic equation is termed *incomplete*; if both $b$ and $c$ are nonzero, the quadratic equation is called *complete*.

**Examples**

$3x^2+8x-5=0$, complete unreduced quadratic equation
$3x^2-5=0$, incomplete unreduced quadratic equation
$x^2-ax=0$, incomplete reduced quadratic equation
$x^2-12x+7=0$, complete reduced quadratic equation.

An incomplete quadratic equation of the form

$$x^2=m \ (m \text{ known})$$

is the simplest case of a quadratic equation and a very important case too, since the solution of every quadratic reduces to this form. The solution of this equation is

$$x=\sqrt{m}$$

Three cases are possible:

(1) If $m=0$, then $x=0$.

(2) If $m$ is a positive number, then its square root $\sqrt{m}$ can have two values: one positive and one negative. Their absolute values are the same. For instance, the equation $x^2=9$ is satisfied by the value $x=+3$ and $x=-3$. In other words, $x$ has two values: $+3$ and $-3$. This is frequently expressed by the double (plus-and-minus) sign in front of the radical: $x=\pm\sqrt{9}$. This notation means that the

expression $\sqrt{9}$ denotes the general absolute value of two roots; in our case, the number 3. The quantity $\sqrt{m}$ may be irrational (see Sec. 91). Note too that the number $m$ itself may be an irrational number. For instance, let it be required to solve the equation

$$x^2=\pi$$

(geometrically, this means finding the length of a side of a square equal to the area of a circle of radius 1). Its root is $x=\sqrt{\pi}$. See Sec. 58 on the extraction of square roots of numbers.

(3) If $m$ is a negative number, then the equation $x^2=m$ (say, $x^2=-9$) cannot have any positive or negative root: this is obvious since either a positive or a negative number, when squared, yields a positive number. We can thus say that the equation $x^2=-9$ has no solution, that is, the number $\sqrt{-9}$ does not exist.

Prior to the introduction of negative numbers we would have been equally justified in saying that the equation $2x+6=4$ has no solutions. But after the introduction of negative numbers this equation became solvable. In the same way, the equation $x^2=-9$ which does not have any solutions in the set of the positive and negative numbers becomes solvable if we introduce new quantities: the square roots of negative numbers. These quantities were first introduced by the Italian mathematician Cardano in the middle of the 16th century in connection with the solution of cubic equations (see Sec. 66). Cardano called these numbers "sophistic". Descartes (in the 1630's) suggested the name "imaginary numbers" which, most unfortunately, has persisted to this day. In contradistinction to the imaginary numbers, the earlier known numbers (positive and negative, including the irrational numbers) came to be called *real numbers*. The sum of a real number and an imaginary number constitutes a so-called *complex number*. This term was introduced by Gauss in 1831. For instance, $2+\sqrt{-3}$ is a complex number. Complex numbers too are sometimes called imaginary numbers. Complex numbers are explained in detail in Secs. 34 et seq.

Having at our disposal imaginary numbers, we can say that the incomplete quadratic equation $x^2=m$ always has two roots. If $m>0$, these roots are real; they have the same absolute value but differ in sign. If $m=0$, both are zero; if $m<0$, they are imaginary.

**93. Solving a Quadratic Equation**

To find the solution of the reduced quadratic

$$x^2+px+q=0$$

it suffices to transpose the constant term to the right member of the equation and to add $\left(\frac{p}{2}\right)^2$ to both members of the equation. Then the left member becomes a perfect square and we get the equivalent equation

$$\left(x+\frac{p}{2}\right)^2=\left(\frac{p}{2}\right)^2-q$$

It differs from the simplest equation $x^2=m$ (see Sec. 92) in aspect alone: $x+\frac{p}{2}$ stands in place of $x$ and $\left(\frac{p}{2}\right)^2-q$ in place of $m$. We find

$$x+\frac{p}{2}=\pm\sqrt{\left(\frac{p}{2}\right)^2-q}$$

Whence

$$\boxed{x=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}} \qquad (1)$$

This formula shows that every quadratic equation has two roots. These roots may be imaginary if $\left(\frac{p}{2}\right)^2<q$. It may also happen that both roots of the quadratic are the same; this occurs when $\left(\frac{p}{2}\right)^2=q$.

Formula (1) is a particularly convenient form to use when $p$ is an even integer.

**Example 1.** $x^2-12x-28=0$.

Here $p=-12$, $q=-28$;

$x=6\pm\sqrt{6^2+28}=6\pm\sqrt{64}=6\pm 8$,

$x_1=6+8=14$,

$x_2=6-8=-2$

**Example 2.** $x^2+12x+10=0$.

$x=-6\pm\sqrt{36-10}=-6\pm\sqrt{26}$,

$x_1=-6+\sqrt{26}\approx -0.9$, $x_2=-6-\sqrt{26}\approx -11.1$

**Example 3.** $x^2-2mx+m^2-n^2=0$.

$$x=m\pm\sqrt{m^2-(m^2-n^2)}=m\pm\sqrt{n^2}=m\pm n,$$

$$x_1=m+n,\ x_2=m-n$$

**Note.** In Example 2, both roots are negative real numbers, but irrational (Sec. 91). The square roots obtained in solving quadratics may be extracted by means of computations (see Sec. 58) or by using tables. Unfortunately, most problem books give exercises in quadratic equations that are specially constructed so that the roots are extracted exactly. In practical situations, this occurs very rarely. For this reason we strongly advise the student to rid himself of the fear of irrational solutions instilled by such problem books.

When $p$ is not an even integer, it is preferable, when solving reduced quadratics, to use the more general formula (3) given below and assume $a=1$ (see Example 5 below).

The unreduced complete quadratic

$$ax^2+bx+c=0 \tag{2}$$

may be solved by the formula

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}} \tag{3}$$

It is obtained from (1) by dividing both members of the unreduced equation (2) by $a$.

**Example 4.** $3x^2-7x+4=0$, $(a=3,\ b=-7,\ c=4)$.

$$x=\frac{7\pm\sqrt{7^2-4\cdot3\cdot4}}{2\cdot3}=\frac{7\pm\sqrt{1}}{6},$$

$$x_1=\frac{7+1}{6}=\frac{4}{3},\ x_2=\frac{7-1}{6}=1$$

**Example 5.** $x^2+7x+12=0$, $(a=1,\ b=7,\ c=12)$.

$$x=\frac{-7\pm\sqrt{49-4\cdot12}}{2},$$

$$x_1=-3,\ x_2=-4$$

**Example 6.** $0.60x^2+3.2x-8.4=0$.

$$x\approx\frac{-3.2\pm\sqrt{(-3.2)^2-4\cdot0.60\cdot(-8.4)}}{2\cdot0.60},$$

$$x_1\approx\frac{-3.2+5.5}{2\cdot0.60}\approx1.9,\quad x_2\approx\frac{-3.2-5.5}{2\cdot0.60}\approx-7.2$$

In Example 6, the coefficients are assumed to be approximate numbers, as is evident from the fact that we write $0.60x^2$ (and not $0.6x^2$). Therefore, it is advisable to perform the operations by the short-cut method given in Secs. 47 and 48. At any rate, note particularly that by the rules of the indicated sections, only two exact significant digits can be obtained. Note that these results are correct to 0.1, but this does not mean that by putting them in the left member of the equation we get a number equal to zero to within 0.1. On the contrary, substituting into the left member, say, the value $x=1.9$, we get

$$0.60\cdot 1.9^2+3.2\cdot 1.9-8.4\approx -0.2$$

But if the value of $x$ is increased by 0.1 and we take $x=2.0$, then we have

$$0.60\cdot 2.0^2+3.2\cdot 2.0-8.4\approx 0.4$$

Thus, for $x=1.9$ the left member was negative; for $x=2.0$ it is positive. This means that it is equal to zero for some value of $x$ lying between 1.9 and 2.0. Consequently, taking $x=1.9$ we err by not more than 0.1. This is what we mean when we say that the root is equal to 1.9 to within an accuracy of 0.1.

If $b$ is an even number, it is best to give the general formula in the form

$$\boxed{x=\frac{-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-ac}}{a}}$$

**Example 7.** $3x^2-14x-80=0$.

$$x=\frac{7\pm\sqrt{7^2+3\cdot 80}}{3}=\frac{7\pm\sqrt{289}}{3}=\frac{7\pm 17}{3},$$

$$x_1=8,\quad x_2=-\frac{10}{3}$$

This formula is also convenient when the coefficients $a$, $b$, $c$, are literal expressions.

**Example 8.** $ax^2-2(a+b)x+4b=0$.

$$x=\frac{a+b\pm\sqrt{(a+b)^2-4ab}}{a}=\frac{a+b\pm\sqrt{a^2-2ab+b^2}}{a}=\frac{a+b\pm(a-b)}{a},$$

$$x_1=2,\quad x_2=2\frac{b}{a}$$

### 94. Properties of the Roots of a Quadratic Equation

The formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

indicates that in the solution of the quadratic equation $ax^2+bx+c=0$ three cases are possible:

(1) $b^2-4ac>0$; two roots are real and distinct;

(2) $b^2-4ac=0$; two roots are real and equal (both are equal to $-\frac{b}{2a}$);

(3) $b^2-4ac<0$; both roots are imaginary.

The expression $b^2-4ac$ which permits us to discriminate between the three cases is termed the *discriminant*.

The signs of the roots, when the roots are real (that is, when $b^2-4ac\geqslant 0$) are best judged on the basis of the following rule of roots.

The sum of the roots of the reduced quadratic equation

$$x^2+px+q=0$$

is equal to the coefficient of the unknown to the first power with sign reversed, that is,

$$x_1+x_2=-p$$

The product of the roots is equal to the constant term, i.e.,

$$x_1x_2=q$$

### 95. Factoring a Trinomial of the Type $ax^2+bx+c$

The quadratic trinomial $ax^2+bx+c$ may be decomposed into first-degree factors in the following manner: solve the quadratic equation $ax^2+bx+c=0$. If $x_1$ and $x_2$ are roots of this equation, then $ax^2+bx+c=a(x-x_1)(x-x_2)$.

**Example 1.** Factor the trinomial $2x^2+13x-24$ into first-degree factors. Solve the equation $2x^2+13x-24=0$. We find the roots $x_1=\frac{3}{2}$, $x_2=-8$. Consequently

$$2x^2+13x-24=2\left(x-\frac{3}{2}\right)(x+8)=(2x-3)(x+8)$$

**Example 2.** Factor $x^2+a^2$; the equation $x^2+a^2=0$ has imaginary roots: $x_1=\sqrt{-a^2}$, $x_2=-\sqrt{-a^2}$ and so it is

impossible to factor $x^2+a^2$ into real factors of the first degree. It can be factored into imaginary factors, however:

$$x^2+a^2=(x+\sqrt{-a^2})(x-\sqrt{-a^2})=(x+ai)(x-ai)$$

($i$ denotes the imaginary number $\sqrt{-1}$)

### 96. Higher-Degree Equations Solvable by Means of Quadratics

Some algebraic equations of higher degree can be solved by reducing them to quadratic equations. The most important cases are the following.

1. The left member of an equation can sometimes readily be decomposed into factors one of which is a polynomial not higher than the second degree. Then we solve the resulting equations by equating each factor to zero separately. The roots thus found will be the roots of the original equation.

**Example 1.** $x^4+5x^3+6x^2=0$.

The polynomial $x^4+5x^3+6x^2$ can readily be factored into $x^2$ and $(x^2+5x+6)$. We solve the equation $x^2=0$ to get two equal roots $x_1=x_2=0$. Solve the equation $x^2+5x+6=0$. Denoting its roots by $x_3$ and $x_4$, we have $x_3=-2$, $x_4=-3$. The roots of the original equation are $x_1=x_2=0$, $x_3=-2$, $x_4=-3$.

**Example 2.** Solve the equation $x^3=8$.

Rewriting it as $x^3-8=0$, we factor the left member: $x^3-8=(x-2)(x^2+2x+4)$. The equation $x-2=0$ yields $x_1=2$, the equation $x^2+2x+4=0$ yields $x_2=-1+\sqrt{-3}$, $x_3=-1-\sqrt{-3}$. Thus, the equation $x^3=8$ has one real root and two imaginary roots. In other words $\sqrt[3]{8}$ has, besides the obvious real value of 2, two imaginary roots (see Sec. 111, Example 3).

2. If the equation is of the form $ax^{2n}+bx^n+c=0$, it can be reduced to a quadratic by introducing a new unknown thus: $x^n=z$.

**Example 3.** $x^4-13x^2+36=0$. Rewriting the equation as $(x^2)^2-13x^2+36=0$ we introduce the new unknown $x^2=z$. The equation then becomes $z^2-13z+36=0$. Its roots are $z_1=9$, $z_2=4$. Now solve the equations $x^2=9$ and $x^2=4$. The first has the roots $x_1=3$, $x_2=-3$, the second, the roots $x_3=2$, $x_4=-2$. The roots of the given equation are: 3, $-3$, 2, $-2$.

It is thus possible to solve any equation of the form $ax^4+bx^2+c=0$, which is called a *biquadratic equation*.

**Example 4.** $x^6-16x^3+64=0$. Representing the equation as $(x^3)^2+16x^3+64=0$, introduce a new unknown: $x^3=z$. This yields the equation $z^2-16z+64=0$, which has two equal roots $z_1=z_2=8$. Solve $x^3=8$ to get (see Example 2) $x_1=2$, $x_2=-1+\sqrt{-3}$, $x_3=-1-\sqrt{-3}$. The other three roots in this case are (since $z_1=z_2$) equal to these three.

## 97. A System of Quadratic Equations in Two Unknowns

The most general form of a second-degree equation in two unknowns is

$$ax^2+bxy+cy^2+dx+ey+f=0$$

where $a$, $b$, $c$, $d$, $e$, $f$ are given numbers or expressions involving known quantities. One equation of the second degree in two unknowns has an infinity of solutions (see Sec. 85).

A system of two equations in two unknowns, one equation being a quadratic equation and the other a first-degree equation, may be solved by the substitution method described in Sec. 86. The expression of one unknown in terms of the other is found from the first-degree equation. Substituting this expression into the second-degree equation, we get an equation in one unknown. In the general case, it is a quadratic equation (see Example 1). However, it may happen that the second-degree terms cancel out; we then have a first-degree equation (see Example 2).

**Example 1.** $x^2-3xy+4y^2-6x+2y=0$, $x-2y=3$.

From the second equation we find $x=3+2y$. Substituting this expression into the first equation, we get

$$(3+2y)^2-3(3+2y)y+4y^2-6(3+2y)+2y=0$$

Solving this equation, we find

$$9+12y+4y^2-9y-6y^2+4y^2-18-12y+2y=0,$$

$$2y^2-7y-9=0,$$

$$y=\frac{7\pm\sqrt{49+72}}{4},$$

$$y_1=\frac{9}{2}, \quad y_2=-1$$

Insert the values thus found: $y_1 = -\frac{9}{2}$, $y_2 = -1$ into the expression $x = 3 + 2y$ to get $x_1 = 12$, $x_2 = 1$.

**Example 2.** $x^2 - y^2 = 1$, $x + y = 2$.

From the second equation we find $y = 2 - x$. Putting this expression into the first equation, we get $x^2 - (2-x)^2 = 1$. After collecting terms, second-degree terms cancel out, and we have $-4 + 4x = 1$, whence $x = \frac{5}{4}$. Putting this value into the expression $y = 2 - x$, we find $y = \frac{3}{4}$.

A system of two quadratic equations in two unknowns can be solved thus: if one of the equations does not contain the term $ax^2$ (or the term $cy^2$), then use the substitution method expressing $x$ (or $y$) of this equation in terms of $y$ (or $x$); but if both equations involve terms like $ax^2$ and $cy^2$, then first apply the method of addition or subtraction (Sec. 85) so as to obtain an equation without $ax^2$ or $cy^2$. Then apply the substitution method. After the elimination process we have an equation in one unknown (generally speaking, of the fourth degree). Only in exceptional cases does it reduce to a quadratic equation, but these cases are encountered rather frequently in the solution of geometric problems.

**Example 3.**

$$x^2 + xy + 2y^2 = 74, \quad 2x^2 + 2xy + y^2 = 73$$

Both equations have terms involving $x^2$ and terms involving $y^2$. So first use the addition-or-subtraction method to obtain an equation without, say, $y^2$.

$$\begin{array}{r|c|l} 2x^2 + 2xy + y^2 = 73 & 2 & - \quad 4x^2 + 4xy + 2y^2 = 146 \\ x^2 + xy + 2y^2 = 74 & & \quad x^2 \mp xy \mp 2y^2 = -74 \\ \hline & & 3x^2 + 3xy \qquad = 72 \end{array}$$

From this equation we find the expression of $y$ in terms of $x$

$$y = \frac{24 - x^2}{x}$$

Then we substitute this expression into one of the given equations, say the first, to get

$$x^2 + x\frac{24 - x^2}{x} + 2\frac{(24 - x^2)^2}{x^2} = 74$$

Simplifications yield

$$x^4+24x^2-x^4+1152-96x^2+2x^4=74x^2,$$
$$2x^4-146x^2+1152=0,$$
$$x^4-73x^2+576=0$$

We have a biquadratic equation (see Sec. 96, Example 3). Putting $x^2=z$, we reduce it to the equation $z^2-73z+576=0$ which yields

$$z=\frac{73\pm\sqrt{73^2-4\cdot576}}{2}=\frac{73\pm\sqrt{3025}}{2}=\frac{73\pm55}{2},$$
$$z_1=64, \quad z_2=9$$

The first solution gives us $x_1=8$, $x_2=-8$, the second, $x_3=3$, $x_4=-3$. Putting the values $x_1$, $x_2$, $x_3$, $x_4$ into the expression $y=\frac{24-x^2}{x}$, we get the corresponding values of $y$:

$$y_1=-5, \quad y_2=+5, \quad y_3=+5, \quad y_4=-5$$

In solving systems of second-degree equations artificial techniques often yield faster and more elegant results.

### 98. Complex Numbers

The development of algebra (Sec. 66) called for numbers of a new kind besides the familiar positive and negative numbers. They received the name *complex numbers*.

A complex number has the form $a+bi$, where $a$ and $b$ are real numbers and $i$ is a new kind of number called the *imaginary unit*. "Imaginary" numbers (see Sec. 92) constitute a special subset of the complex numbers (when $a=0$). On the other hand, the real numbers (positive and negative) also constitute a special subset of the complex numbers (when $b=0$).

Let us call the real number $a$ the *abscissa* of the complex number $a+bi$; the real number $b$, the *ordinate* of the complex number $a+bi$. The basic property of the number $i$ is that the product $i\cdot i$ is equal to $-1$, that is

$$i^2=-1 \qquad (1)$$

For a long time, no physical quantities could be found that obeyed the same rules as those involving complex numbers, say Rule (1). Whence the name "imaginary unit",

"imaginary number", etc. At present we know many such physical quantities, and complex numbers are extensively employed not only in mathematics but also in physics and engineering (theory of elasticity, electrical engineering, aerodynamics, etc.).

Below (Sec. 104) a geometric interpretation of complex numbers is given, but first the rules for operating with them are considered (Secs. 100-103). Here, the question of the geometric or physical meaning of the number $i$ is not considered because it differs in various spheres of science.

The rule for each operation involving complex numbers is derived from the definition of the operation. The definitions of the operations involving complex numbers were not devised arbitrarily but were established to ensure consistency with the rules for operating with real numbers (cf. Sec. 34). The point is that complex numbers must not be considered separately from the real numbers.

### 99. Basic Conventions Concerning Complex Numbers

1. The real number $a$ can also be written as $a+0\cdot i$ (or $a-0\cdot i$).

**Examples.** The notation $3+0\cdot i$ means the same as 3. The notation $-2+0\cdot i$ means $-2$. The notation $\frac{3\sqrt{2}}{2}+0\cdot i$ is equivalent to $\frac{3\sqrt{2}}{2}$.

**Note.** We have similar instances in ordinary arithmetic where the fraction $\frac{5}{1}$ denotes the same thing as 5. The notation 002 is simply 2, etc.

2. A complex number of the form $0+bi$ is called a pure imaginary. The notation $bi$ is the same as $0+bi$.

3. Two complex numbers $a+bi$, $a'+b'i$ are considered equal if they have equal abscissas and ordinates, that is, if $a=a'$, $b=b'$. Otherwise these complex numbers are not equal. This definition is suggested by the following reasoning. If, say, we could have an equation like $2+5i=8+2i$, then by the rules of algebra we would get $i=2$, whereas $i$ cannot be a real number.

**Note.** We have not yet defined the addition of complex numbers, and so, strictly speaking, we cannot yet assert that the number $2+5i$ is the sum of the numbers 2 and $5i$. It

would be more correct to say that we have here a *pair of real numbers* 2 (abscissa) and 5 (ordinate); these numbers generate a new kind of number which we agree to denote by $2+5i$.

### 100. Addition of Complex Numbers

**Definition.** The sum of the complex numbers $a+bi$ and $a'+b'i$ is the complex number $(a+a')+(b+b')i$.

This definition is suggested by the rules for operating on ordinary polynomials.

**Example 1.** $(-3+5i)+(4-8i)=1-3i$.

**Example 2.** $(2+0i)+(7+0i)=9+0i$. Since (Sec. 99) the notation $2+0i$ means the same as 2, etc., the operation performed is in agreement with ordinary arithmetic $(2+7=9)$.

**Example 3.** $(0+2i)+(0+5i)=0+7i$, that is to say, (Sec. 99), $2i+5i=7i$.

**Example 4.** $(-2+3i)+(-2-3i)=-4$.

In Example 4 the sum of two complex numbers is equal to a real number. Two complex numbers like $a+bi$ and $a-bi$ are called *conjugate complex numbers*. The sum of the two conjugate complex numbers is equal to the real number $2a$. Note that the sum of two nonconjugate complex numbers can also be a real number, as witness $(3+5i)+(4-5i)=7$.

**Note.** Now that the operation of addition has been defined, we can consider the complex number $a+bi$ as the sum of the numbers $a$ and $bi$. Thus, the number 2 (which we agreed could be written as $2+0i$) and the number $5i$ (which, by Sec. 99, means the same as $0+5i$) yield a sum equal (according to the definition) to $2+5i$.

### 101. Subtraction of Complex Numbers

**Definition.** The difference between the complex number $a+bi$ (minuend) and $a'+b'i$ (subtrahend) is the complex number $(a-a')+(b-b')i$.

**Example 1.** $(-5+2i)-(3-5i)=-8+7i$.

**Example 2.** $(3+2i)-(-3+2i)=6+0i=6$.

**Example 3.** $(3-4i)-(3+4i)=-8i$.

**Note.** The subtraction of complex numbers can be defined as an operation inverse to addition. Namely, we seek a comp-

lex number $x+yi$ (difference) such that $(x+yi)+(a'+b'i)=a+bi$. By the definition of Sec. 100, we have

$$(x+a')+(y+b')i=a+bi$$

By the condition for the equality of complex numbers (Sec. 99),

$$x+a'=a, \quad y+b'=b$$

From these equations we find $x=a-a'$, $y=b-b'$

### 102. Multiplication of Complex Numbers

The definition of the multiplication of complex numbers is devised so that (1) the numbers $a+bi$ and $a'+b'i$ can be multiplied as algebraic binomials and (2) the number $i$ has the property $i^2=-1$. By virtue of Requirement 1, the product $(a+bi)(a'+b'i)$ must be equal to $aa'+(ab'+ba')i+bb'i^2$, and by virtue of Requirement 2, this expression must equal $(aa'-bb')+(ab'+ba')i$. Accordingly we have the following definition.

**Definition.** The product of the complex numbers $a+bi$ and $a'+b'i$ is the complex number

$$(aa'+bb')+(ab'+ba')i \tag{1}$$

**Note 1.** The equality $i^2=-1$ was in the nature of a requirement prior to establishment of the rules for multiplication. Now however it follows from the definition. The point is that the notation $i^2$, that is, $i \cdot i$, is equivalent (Sec. 99) to the notation $(0+1 \cdot i)(0+1 \cdot i)$. Here, $a=0$, $b=1$, $a'=0$, $b'=1$. We have $aa'-bb'=-1$, $ab'+ba'=0$, so that the product is $-1+0i$, that is, $-1$.

**Note 2.** In practical situations there is no need to use formula (1). One can multiply the given numbers as binomials and then put $i^2=-1$.

**Example 1.** $(1-2i)(3+2i)=3-6i+2i-4i^2=3-6i+2i+4=7-4i$.

**Example 2.** $(a+bi)(a-bi)=a^2+b^2$.

Example 2 shows that the product of conjugate complex numbers is a real positive number. *

* But the product of two nonconjugate complex numbers can also be a positive real number; for example $(2+3i)(4-6i)=26$ (see Sec. 100). Now if both the sum and the product of two complex numbers are real numbers, then these complex numbers must be conjugate.

### 103. Division of Complex Numbers

In accordance with the definition of the division of real numbers we give the following definition.

**Definition.** To divide a complex number $a+bi$ (dividend) by a complex number $a'+b'i$ (divisor) means to find a number $x+yi$ (quotient) such that, when multiplied by the divisor, it yields the dividend.

If the divisor is nonzero, then division is always possible and the quotient is unique (for proof see Note 2). A convenient practical way to find the quotient is this.

**Example 1.** Find the quotient of $(7-4i):(3+2i)$.

Writing the fraction $\frac{7-4i}{3+2i}$, multiply both terms by number $3-2i$, which is the conjugate of $3+2i$ (see Sec. 102, Example 1), to get

$$\frac{(7-4i)(3-2i)}{(3+2i)(3-2i)}=\frac{13-26i}{13}=1-2i$$

Example 1 of the preceding section is a check.

**Example 2.** $\frac{-2+5i}{-3-4i}=\frac{(-2+5i)(-3+4i)}{(-3-4i)(-3+4i)}=\frac{-14-23i}{25}=-0.56-0.92i$.

**Example 3.** $\frac{-6+21i}{4-14i}=-\frac{3}{2}$. The easiest way here is to divide both terms by $(-2+7i)$.

Doing as we did in Examples 1 and 2, we find the general formula

$$(a+bi):(a'+b'i)=\frac{aa'+bb'}{a'^2+b'^2}+\frac{a'b+b'a}{a'^2+b'^2}i \qquad (1)$$

In order to prove that the right member of (1) is really the quotient, multiply it by $a'+b'i$ to get $a+bi$.

**Note 1.** We could take formula (1) as the definition of division (cf. the definitions in Secs. 100 and 101).

**Note 2.** Formula (1) can also be derived as follows. By the definition we must have $(a'+b'i)(x+yi)=a+bi$. Hence (Sec. 99) the following two equations must be satisfied:

$$a'x-b'y=a, \quad b'x+a'y=b \qquad (2)$$

This system has a unique solution:

$$x=\frac{aa'+bb'}{a'^2+b'^2}, \quad y=\frac{a'b+b'a}{a'^2+b'^2}$$

if $\frac{a'}{b'}\neq-\frac{b'}{a'}$ (Sec. 87), that is, if $a'^2+b'^2\neq 0$.

It remains to consider the case $a'^2+b'^2=0$. This case (the numbers $a'$ and $b'$ are real!) is only possible when $a'=0$ and $b'=0$, that is when the divisor $a'+b'i$ is zero. If however the dividend $a+bi$ is also zero, then the quotient is indeterminate (see Sec. 37, Item 2.) But if the dividend is not zero, then the quotient does not exist (we say it is equal to infinity) (cf. Sec. 37, Item 3).

### 104. Complex Numbers Depicted in the Plane

Real numbers may be depicted by points on a straight line, as shown in Fig. 2 where point $A$ denotes number 4 and point $B$ number $-5$. These same numbers can also be indicated by the line-segments $OA$, $OB$, which indicate both length and direction.

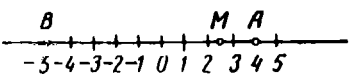



Fig. 2.

Every point $M$ of the number line depicts some real number (rational if the line segment $OM$ is commensurable with the unit length, and irrational if it is incommensurable). Thus, there is no place for complex numbers on the number line.

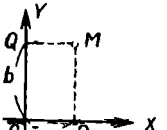



Fig. 3

But complex numbers can be depicted on a "number plane". To do this we choose a rectangular system of coordinates in the plane (see Sec. 211) with the same scale on both axes (Fig. 3). The complex number $a+bi$ is depicted by a point $M$, whose abscissa $x$ (in Fig. 3, $x=OP=QM$) is equal to the abscissa $a$ of the complex number and the ordinate $y$ ($OQ=PM$) is equal to the ordinate $b$ of the complex number.

**Examples.** In Fig. 4, point $A$ with abscissa $x=3$ and ordinate $y=-5$ depicts the complex number $3+5i$. Point $E$ depicts the complex number $2+6i$; point $C$ is the complex number $-6-2i$; point $D$ denotes the complex number $2-6i$.

Real numbers (in complex form they look like this $a+0i$) are depicted by points on the $X$-axis and pure ima-

ginary numbers (of the form $0+bi$), by points on the $Y$-axis.

**Examples.** Point $K$ in Fig. 4 depicts the real number 6 (or, what is the same thing, the number $6+0i$), point $L$ depicts the pure imaginary number $3i$ (i. e., $0+3i$); point $N$, the pure imaginary $-4i$ (i. e., $0-4i$). The origin of the coordinates indicates the number 0 (which is $0+0i$).

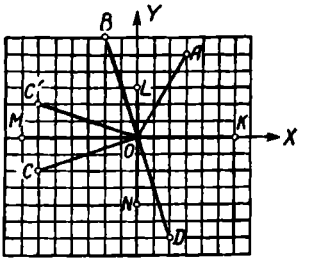


Fig. 4.

Conjugate complex numbers are depicted by a pair of points symmetric about the axis of abscissas ($X$-axis); thus, the points $C$ and $C'$ in Fig. 4 indicate the conjugate numbers $-6-2i$ and $-6+2i$.

Complex numbers can also be indicated by line-segments (vectors) issuing from the point $O$ and terminating at the appropriate point of the number plane. Thus, the complex number $-2+6i$ may be described not only by the point $B$ (Fig. 4), but also by the vector $OB$. The complex number $-6-2i$ is denoted by the vector $OC$, etc.

**Note.** When we call some line-segment a *vector* we mean that there are two essential things about the line: it has length and it has a certain *direction*. Two vectors are considered the same (equal) only when they have the same length and the same direction.

### 105. The Modulus and Argument of a Complex Number

The length of the vector depicting a complex number is called the *modulus* of the complex number. The modulus of any complex number not equal to zero is a positive number.

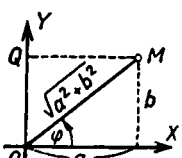


Fig. 5.

The modulus of the complex number $a+bi$ is symbolized as $|a+bi|$ and also by the letter $r$. From the drawing (Fig. 5) it is clear that

$$r=|a+bi|=\sqrt{a^2+b^2} \qquad (1)$$

The modulus of a real number coincides with its absolute value. The conjugate complex numbers $a+bi$ and $a-bi$ have the same modulus.

**Example 1.** The modulus of the complex number $3+5i$ (that is, the length of the vector $OA$ in Fig. 4) is equal to $\sqrt{3^2+5^2}=\sqrt{34}\approx 5.83$.

**Example 2.** $|1+i|=\sqrt{1^2+1^2}=\sqrt{2}\approx 1.41$.

**Example 3.** $|-3+4i|=5$.

**Example 4.** The modulus of the number $-7$ (that is, $-7+0i$) is the length of the vector $OM$ (Fig. 4). This length is expressed by the positive number 7, or

$$|-7+0i|=\sqrt{(-7)^2+0^2}=7$$

**Example 5.** The modulus of the number $-4i$ (length of vector $ON$, Fig. 4) is 4.

**Example 6.** The modulus of the number $-6-2i$ (length of vector $OC$, Fig. 4) is equal to $\sqrt{40}\approx 6.32$. The modulus of the number $-6+2i$ (length of vector $OC'$, Fig. 4) is also equal to $\sqrt{40}$. Two conjugate complex numbers always have equal moduli.

The angle $\varphi$ between the axis of abscissas and the vector $OM$ describing the complex number $a+bi$ is called the *argument* of the complex number $a+bi$. In Fig. 6, the vector $OM$ depicts the complex number $-3-3i$. The angle $XOM$ is the argument of this complex number.

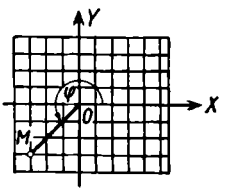


Fig. 6.

Every nonzero complex number (for the number 0 the argument is indeterminate) has an infinite number of arguments that differ by an integral number of complete rotations (that is, by $360°k$, where $k$ is any integer). Thus, the arguments of the complex numbers $-3-3i$ are all angles of the form $225°\pm 360°k$, for instance, $225°+360°=585°$, $225°-360°=-135°$.

The argument $\varphi$ is connected with the coordinates of a complex number $a+bi$ by the following formulas (see Fig. 5):

$$(1)\ \tan\varphi=\frac{b}{a},\quad (2)\ \cos\varphi=\frac{a}{\sqrt{a^2+b^2}},\qquad (4)$$

$$(3)\ \sin\varphi=\frac{b}{\sqrt{a^2+b^2}}$$

However, none of these formulas, taken separately, permits finding the argument from the abscissa and ordinate (see examples).

**Example 7.** Find the argument of the complex number $-3-3i$.

By formula (2) $\tan\varphi=\frac{-3}{-3}=1$. This condition is satisfied by an angle of 45° and also by an angle of 225°. But the 45° angle is not the argument of the number $-3-3i$ (Fig. 6). The correct answer is $\varphi=225°$ (or $-135°$, or 585°, etc.). This result is obtained if we note that the abscissa and the ordinate of the given complex number are negative. This means that point $M$ lies in the third quadrant.

**Alternative method.** Use formula (3) to find $\cos\varphi=\frac{-1}{\sqrt{2}}$. Formula (4) shows that $\sin\varphi$ is also negative. Hence the angle $\varphi$ belongs to the third quadrant so that $\varphi=225°\pm360°k$.

**Example 8.** Find the argument of the complex number $-2+6i$. We find $\tan\varphi=\frac{6}{-2}=-3$. Since the abscissa is negative and the ordinate is positive, the angle $\varphi$ lies in the second quadrant. Using tables we find $\varphi\approx180°-72°=108°$. See Fig. 4 where point $B$ depicts $-2+6i$.

The least (in absolute value) argument is called the *principal value* of the argument. Thus, for the complex numbers $-3-3i$, $2i-5i$, the principal values of the argument are $-135°$, $+90°$, $-90°$.

The argument of a positive real number has the principal value 0°; for negative numbers, the principal value of the argument is taken to be 180° (and not $-180°$).

In the case of conjugate complex numbers, the principal values of the argument have the same absolute values but opposite signs. Thus, the principal values of the argument of the numbers $-3+3i$ and $-3-3i$ are 135° and $-135°$.

### 106. Trigonometric Form of a Complex Number

Abscissa $a$ and ordinate $b$ of the complex number $a+bi$ are expressed in terms of the modulus $r$ and the argument $\varphi$ (see Fig. 5) by the formulas

$$a=r\cos\varphi,\quad b=r\sin\varphi$$

Therefore, any complex number can be represented in the form $r(\cos\varphi+i\sin\varphi)$ where $r\geqslant0$.

This expression is termed the *normal trigonometric form* or, simply, the trigonometric form of a complex number.

**Example 1.** Represent the complex number $-3-3i$ in the normal trigonometric form. We have (Sec. 105)

$$r=\sqrt{(-3)^2+(-3)^2}=3\sqrt{2}$$

Consequently

$$-3-3i=3\sqrt{2}(\cos(-135°)+i\sin(-135°))$$

or

$$-3-3i=3\sqrt{2}(\cos 225°+i\sin 225°)$$

and so forth.

**Example 2.** For the complex number $-2+6i$ we have

$$r=\sqrt{(-2)^2+6^2}=\sqrt{40}$$

and (Sec. 105, Example 2) $\varphi=108°$. Hence, the normal trigonometric form of the number $-2+6i$ is

$$\sqrt{40}(\cos 108°+i\sin 108°)$$

**Example 3.** The normal trigonometric form of the number 3 is $3(\cos 0°+i\sin 0°)$ or, in general form,

$$3(\cos 360°k+i\sin 260°k)$$

**Example 4.** The normal trigonometric form of the number $-3$ is $3(\cos 180°+i\sin 180°)$ or

$$3[\cos(180°+360°k)+i\sin(180°+360°k)]$$

**Example 5.** The normal trigonometric form of the imaginary unit $i$ is $\cos 90°+i\sin 90°$ or

$$\cos(90°+360°k)+i\sin(90°+360°k)$$

Here $r=1$.

**Example 6.** The normal trigonometric form of the number $-i$ is $\cos(-90°)+i\sin(-90°)$ or

$$\cos(-90°+360°k)+i\sin(-90°+360°k)$$

Here $r=1$.

In contrast to the trigonometric form, an expression like $a+bi$ is called the *algebraic* or *coordinate* (Cartesian) form of the complex number.

**Example 7.** Represent the complex number $2[\cos(-40°)+i\sin(-40°)]$ in algebraic form.

Here $r=2$, $\varphi=-40°$. By formulas (3) and (4) of the preceding section,

$$a=r\cos\varphi=2\cos(-40°)\approx 2\cdot 0.766=1.532,$$
$$b=r\sin\varphi=2\sin(-40°)\approx 2\cdot(-0.643)=-1.286$$

The algebraic form of the given number is (approximately) $1.532-1.286i$.

**Example 8.** Represent in algebraic form the number $3(\cos 270^\circ + i \sin 270^\circ)$. Since $\cos 270^\circ = 0$, $\sin 270^\circ = -1$, the given number is equal to $-3i$.

**Example 9.** If $r(\cos \varphi + i \sin \varphi)$ is one of two conjugate complex numbers, then the other can be represented in the form $r[\cos(-\varphi) + i \sin(-\varphi)]$ or in the form $r(\cos \varphi - i \sin \varphi)$; incidentally, the latter expression is not the normal form.

## 107. Geometric Meaning of Addition and Subtraction of Complex Numbers

Let the vectors $OM$ and $OM'$ (Fig. 7) depict the complex numbers $z = x + yi$ and $z' = x' + y'i$. From the point $M$ draw the vector $MK$ equal to $OM'$ (that is, having the same length and the same direction as $OM'$; see Sec. 104, note). Then the vector $OK$ gives the sum of the given complex numbers.*

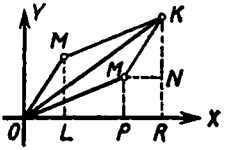


Fig. 7.

The vector $OK$ thus constructed is termed the *geometric sum* (or, briefly, the sum) of the vectors $OM$ and $OM'$ (the name "sum" is due to the fact that it arises by analogy with the combining of velocities of moving bodies, of forces applied to a point, and of many other physical quantities).

Thus, the sum of two complex numbers is given by the sum of the vectors depicting the separate summands.

The length of side $OK$ of the triangle $OMK$ is less than the sum and greater than the difference of the lengths $OM$ and $MK$. Therefore

$$\| z | - | z' \| \leqslant | z + z' | \leqslant | z | + | z' |$$

The equality is valid only when the vectors $OM$ and $OM'$ have the same (Fig. 8) or opposite directions (Fig. 9). In the former case, $|OM| + |OM'| = |OK|$, that is $|z + z'| = |z| + |z'|$. In the latter case $|z + z'| = \| z | - | z' \|$.

---

* Indeed, the triangles $OM'L$ and $MKN$ are equal. Hence, $x' = OL = MN = PR$, $y' = LM' = NK$. Consequently, the abscissa $OR = OP + PR = x + x'$, and the ordinate $RK = y + y'$.

**Example 1.** Let $z=4+3i$, $z'=5+12i$. Then

$$|z|=\sqrt{4^2+3^2}=5,\quad |z'|=\sqrt{5^2+12^2}=13,\quad z+z'=9+15i,$$
$$|z+z'|=\sqrt{9^2+15^2}=\sqrt{306}$$

We have $13-5<\sqrt{306}<13+5$, $8<\sqrt{306}<18$.

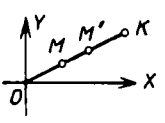



Fig. 8.

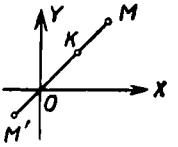



Fig. 9.

**Example 2.** Let $z=4+3i$, $z'=8+6i$.
These complex numbers have the same argument (36°52′), the corresponding vectors are in the same direction. Here

$$|z|=5,\quad |z'|=10,\quad z+z'=12+9i,$$
$$|z+z'|=\sqrt{12^2+9^2}=15$$

We have $10-5<15=10+5$.

**Example 3.** Let $z=8-6i$, $z'=-12+9i$.
These complex numbers are depicted by vectors of opposite direction (their arguments are equal to 323°08′ and 143°08′). Here

$$|z|=10,\quad |z'|=15,\quad z+z'=-4+3i,\quad |z+z'|=5$$

We have

$$15-10=5<15+10$$

The *sum of three or more complex numbers* can also be represented as the sum of vectors ($OM$, $OM'$, $OM''$ in Fig. 10) depicting the separate summands, that is to say, the vector $OK$ which completes the polygonal line $OMSK$ (the vector $MS$ is equal to the vector $OM'$; the vector $SK$ equals the vector $OM''$). The summands may be taken in any order; the polygonal lines will be different but their extremities will coincide. Since $OK$ is not longer than the polygonal line $OMSK$, it follows that

$$|z+z'+z''|\leqslant|z|+|z'|+|z''|$$

The equality is valid only when all summands have the same direction.

The difference between the complex numbers $a+bi$ and $a'+b'i$ is equal to the sum of the numbers $a+bi$ and $-a'-b'i$. The latter summand has the same modulus as

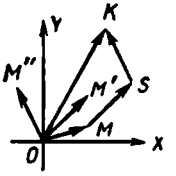


Fig. 10

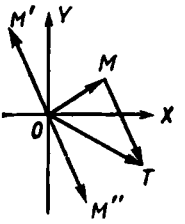


Fig. 11

$a'+b'i$ but is opposite in direction. Therefore, the difference between two complex numbers represented by the vectors $OM$ and $OM'$ (Fig. 11) is depicted as the sum of the vectors $OM$ and $OM''$ (as the vector $OT$).

### 108. Geometric Meaning of Multiplication of Complex Numbers

Let two complex numbers $z$ and $z'$ be depicted by the vectors $OM$ and $OM'$ (Fig. 12). Write down the factors in trigonometric form and compute the product:

$$\begin{aligned} zz' &= r(\cos\varphi + i\sin\varphi)\cdot r'(\cos\varphi' + i\sin\varphi') \\ &= rr'[(\cos\varphi\cos\varphi' - \sin\varphi\sin\varphi') + i(\sin\varphi\cos\varphi' \\ &\quad + \cos\varphi\sin\varphi')] \end{aligned}$$

That is (Sec. 194)

$$zz' = rr'[\cos(\varphi+\varphi') + i\sin(\varphi+\varphi')] \qquad (1)$$

The modulus of the product (it is depicted by the vector $OL$) is $rr'$, and the argument of the product is equal to $\varphi+\varphi'$. In other words, *to multiply two complex numbers, multiply their moduli and add their arguments.*

This rule holds true for any number of factors.

**Example 1.** Take the complex numbers depicted by the vectors $OM$ and $OM'$ in Fig. 12. Their moduli are $|OM| = \frac{3}{2}$ and $|OM'| = 2$, and the arguments are $\angle XOM = 20°$ and $\angle XOM' = 30°$. The modulus of the product depicted by the

vector $OL$ is $\frac{3}{2}\cdot 2=3$; the argument of the product (angle $XOL$) is $20°+30°=50°$. We get

$$\frac{3}{2}(\cos 20°+i\sin 20°)\cdot 2(\cos 30°+i\sin 30°)$$

$$=3(\cos 50°+i\sin 50°)$$

**Example 2.**

$$4\sqrt{2}(\cos 45°+i\sin 45°)\cdot\frac{\sqrt{2}}{2}(\cos 135°+i\sin 135°)$$

$$=4(\cos 180°+i\sin 180°)=-4 \text{ (Fig. 13).}$$

The same factors in algebraic form are $4+4i$ and $-\frac{1}{2}+\frac{1}{2}i$. Multiplying them together, we again get $-4$.

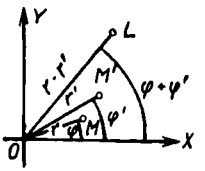



Fig. 12

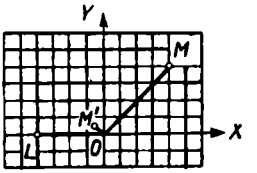



Fig. 13

**Example 3.** Multiply together $2(\cos 150°+i\sin 150°)$, $3[\cos(-160°)+i\sin(-160°)]$ and $0.5(\cos 10°+i\sin 10°)$. The modulus of the product $2\cdot 3\cdot 0.5=3$. The argument of the product $150°-160°+10°=0°$. The product is

$$3(\cos 0°+i\sin 0°)=3$$

**Example 4.** $r(\cos\varphi+i\sin\varphi)\cdot r[\cos(-\varphi)+i\sin(-\varphi)]$ $=r^2(\cos 0°+i\sin 0°)=r^2$.
The product of two conjugate complex numbers is a real number equal to the square of their common modulus.

**Example 5.** $\frac{3}{2}[\cos(-20°)+i\sin(-20°)]\cdot 2[\cos(-30°)$ $+i\sin(-30°)]=3[\cos(-50°)+i\sin(-50°)]$.
Comparing this with Example 1, we see that by replacing the factors with conjugate numbers the product is replaced with its conjugate number. This property is general and can be extended to any number of factors.

**Note 1.** The rules for multiplying real numbers are a special case of the above rule. Thus, in multiplying the numbers $-2$ and $-3$ their arguments (180° and 180°) combine

to form 360° so that the product is the positive number 6 [that is, 6 (cos 360° + $i$ sin 360°)].

**Note 2.** When a complex number $r(\cos\varphi + i\sin\varphi)$ is multiplied by the imaginary unit $i$ (the modulus of which is 1 and the argument +90°), the modulus of the product remains equal to $r$. The argument however is increased by 90°, which means the vector of the factor is rotated through +90° without changing its length. In particular, multiplication of 1 (the vector $OA$ in Fig. 14) by $i$ is represented by a rotation of the vector $OA$ through 90° to position $OB$, while multiplication of $i$ by $i$ is represented by a rotation of $OB$ through 90° to the position $OC$. But the vector $OC$ is depicted as $-1$. Therefore, $i^2 = -1$. In geometric representation, the number $i$ is no more "imaginary" than the number $-1$.

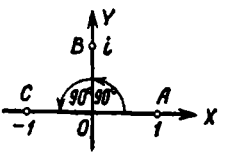


Fig. 14.

## 109. Geometric Meaning of Division of Complex numbers

Division is the inverse of multiplication. Therefore, (see Sec. 108), when dividing complex numbers, divide their moduli (the modulus of the dividend by the modulus of the divisor) and subtract the arguments (the argument of the divisor from the argument of the dividend), or

$$r(\cos\varphi + i\sin\varphi) : r'(\cos\varphi' + i\sin\varphi') = \frac{r}{r'}[\cos(\varphi - \varphi') + i\sin(\varphi - \varphi')] \qquad (1)$$

**Example 1.** $2(\cos 30° + i\sin 30°) : 6(\cos 45° + i\sin 45°) = \frac{1}{3}[\cos(-15°) + i\sin(-15°)]$.

**Example 2.** $-4 : 4\sqrt{2}(\cos 45° + i\sin 45°) = 4(\cos 180° + i\sin 180°) : 4\sqrt{2}(\cos 45° + i\sin 45°) = \frac{1}{\sqrt{2}}(\cos 135° + i\sin 135°)$. Cf. Example 2 of the preceding section.

In algebraic form

$$-4 : (4 + 4i) = \frac{-1}{1+i} = \frac{-1(1-i)}{(1+i)(1-i)} = \frac{-1+i}{2}$$

**Example 3.** Divide 1 by the complex number $r(\cos\varphi + i\sin\varphi)$. The dividend can be written as $1(\cos 0° + i\sin 0°)$.

By formula (1) the quotient will be $\frac{1}{r}[\cos(-\varphi)+i\sin(-\varphi)]$.

$$1:r(\cos\varphi+i\sin\varphi)=\frac{1}{r}[\cos(-\varphi)+i\sin(-\varphi)] \quad (2)$$

Geometric construction: describe a circle of radius 1 with centre at $O$. Let $|r|>1$, that is, the point $M$ (Fig. 15) depicting the divisor lies outside the circle. Draw the tangent $MT$; from point $T$ draw the perpendicular $TM'$ to $OM$. The point $L$ which is symmetric with $M'$ about the axis of abscissas depicts the quotient. Indeed, $|OL|=|OM'|$ and from the right triangle $OTM$, in which $TM'$ is the altitude,

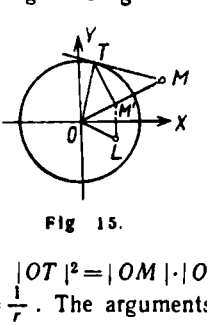



Fig 15.

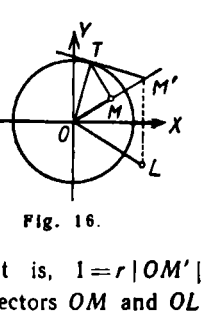



Fig. 16.

we find $|OT|^2=|OM|\cdot|OM'|$; that is, $1=r|OM'|$ or $|OM'|=\frac{1}{r}$. The arguments of the vectors $OM$ and $OL$ are obviously equal in magnitude and opposite in sign.

For the case $|r|<1$ see construction in Fig. 16.

From formula (2) it follows that division of 1 by a complex number with modulus $r=1$ yields a complex number that is conjugate to the divisor.

**Example 4.** $2[\cos(-30°)+i\sin(-30°)]:6[\cos(-45°)+i\sin(-45°)]=\frac{1}{3}(\cos 15°+i\sin 15°)$.

Comparing this with Example 1, we see that replacing the dividend and divisor by conjugate numbers replaces the quotient by its conjugate number. Formula (1) shows that this property is general.

### 110. Raising a Complex Number to an Integral Power

According to Sec. 108,

$$[r(\cos\varphi+i\sin\varphi)]^2=r^2(\cos 2\varphi+i\sin 2\varphi),$$
$$[r(\cos\varphi+i\sin\varphi)]^3=r^3(\cos 3\varphi+i\sin 3\varphi)$$

and generally

$$[r(\cos\varphi + i\sin\varphi)]^n = r^n(\cos n\varphi + i\sin n\varphi) \qquad \text{(A)}$$

where $n$ is a positive integer. Formula (A) is called *De Moivre's theorem* (after Abraham De Moivre, 1667—1754). It is valid for a negative integral exponent $n$ (Sec. 125) and also for $n=0$.

For example, $[r(\cos\varphi + i\sin\varphi)]^{-3}$

$$= \frac{1}{[r(\cos\varphi + i\sin\varphi)]^3} = \frac{1}{r^3(\cos 3\varphi + i\sin 3\varphi)}$$

Consequently (cf. Example 3 of the preceding section),

$$[r(\cos\varphi + i\sin\varphi)]^{-3} = r^{-3}[\cos(-3\varphi) + i\sin(-3\varphi)]$$

To summarize, to raise a complex number to any integral power, raise the modulus to that power and multiply the argument by the exponent of the power. For raising to a fractional power see Sec. 112.

**Example 1.** Raise to the sixth power the number

$$z = 2(\cos 10^\circ + i\sin 10^\circ)$$

We have $z^6 = 2^6(\cos 60^\circ + i\sin 60^\circ) = 32 + 32i\sqrt{3}$.

**Example 2.** Raise to the 20th power the number

$$z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

The modulus of the number $z$ (Sec. 105) is 1, the argument is $-60^\circ$. Hence, the modulus of $z^{20}$ is 1 and the argument is $-1200^\circ = -3\cdot 360^\circ - 120^\circ$. We thus have

$$z^{20} = \cos(-120^\circ) + i\sin(-120^\circ) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

**Example 3.** Find the expression of the cosine and sine of the angle $3\varphi$ in terms of the cosine and sine of the angle $\varphi$.

**Solution.** $\cos 3\varphi + i\sin 3\varphi = (\cos\varphi + i\sin\varphi)^3 = \cos^3\varphi + 3i\cos^2\varphi\sin\varphi + 3i^2\cos\varphi\sin^2\varphi + i^3\sin^3\varphi = \cos^3\varphi - 3\cos\varphi\sin^2\varphi + i(3\cos^2\varphi\sin\varphi - \sin^3\varphi)$

Equating abscissas and ordinates (Sec. 99), we find

$$\cos 3\varphi = \cos^3\varphi - 3\sin^2\varphi\cos\varphi$$

and

$$\sin 3\varphi = 3\cos^2\varphi\sin\varphi - \sin^3\varphi$$

**Example 4.** In the same way we find

$$\cos 4\varphi = \cos^4\varphi - 6\cos^2\varphi\sin^2\varphi + \sin^4\varphi,$$

$$\sin 4\varphi = 4\cos^3\varphi\sin\varphi - 4\cos\varphi\sin^3\varphi$$

and also the general formulas for $\sin n\varphi$, $\cos n\varphi$ (see Sec. 198).

### 111. Extracting the Root of a Complex Number

Extracting the root of a number (see Sec. 23, Item 6) is the inverse of raising a number to a power. Therefore (see preceding section), the modulus of a root (of integral index) of a complex number is obtained by extracting that root of the modulus of the radicand, the argument is obtained by dividing the argument by the index of the root:

$$\sqrt[n]{r(\cos\varphi+i\sin\varphi)}=\sqrt[n]{r}\left(\cos\frac{\varphi}{n}+i\sin\frac{\varphi}{n}\right) \qquad \text{(B)}$$

Here the symbol $\sqrt[n]{r}$ denotes a positive number (the principal root of the modulus).

The $n$th root of any complex number has $n$ distinct values. They all have the same moduli $\sqrt[n]{r}$; the arguments however are obtained from the argument of one of them by successively adding the angle $\frac{1}{n}\cdot 360°$.

Indeed, let $\varphi_0$ be the argument of the radicand. Then $\varphi_0+360°$, $\varphi_0+2\cdot 360°$, etc. are also its arguments. Formula (B) shows that for the argument of the root we can take not only $\frac{\varphi_0}{n}$ but also $\frac{\varphi_0}{n}+\frac{1}{n}\cdot 360°$, $\frac{\varphi_0}{n}+\frac{2}{n}\cdot 360°$, and so on. The corresponding values of the root are not all distinct: the argument $\frac{\varphi_0}{n}+\frac{n}{n}\cdot 360°$, i. e., $\frac{\varphi_0}{n}+360°$ yields the same complex number as the argument $\frac{\varphi_0}{n}$; the argument $\frac{\varphi_0}{n}+\frac{n+1}{n}\times 360°=\frac{\varphi_0}{n}+\frac{1}{n}\,360°+360°$ yields the same complex number as the argument $\frac{\varphi_0}{n}+\frac{1}{n}\cdot 360°$, etc. There will be exactly $n$ distinct values of the root. See examples.

**Example 1.** Extract the square root of $-9i$. The modulus of this number is 9. Hence the modulus of the root is $\sqrt{9}=3$. The argument of the radicand may be taken equal to $-90°$, $-90°+360°$, $-90°+2\cdot 360°$, etc.

In the first case we obtain

$$(-9i)^{\frac{1}{2}}=\sqrt{9}\,[\cos(-45°)+i\sin(-45°)]=\frac{3}{\sqrt{2}}-\frac{3}{\sqrt{2}}i \qquad (1)$$

In the second case

$$(-9i)^{\frac{1}{2}} = \sqrt{9}(\cos 135^\circ + i \sin 135^\circ) = -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i \quad (2)$$

In the third case

$$(-9i)^{\frac{1}{2}} = \sqrt{9}(\cos 315^\circ + i \sin 315^\circ) = \frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}} i$$

which is the same as in the first. Taking $\varphi = -90^\circ + 3 \cdot 360^\circ$, $\varphi = -90^\circ + 4 \cdot 360^\circ$ or $\varphi = -90^\circ - 360^\circ$, $-90^\circ - 2 \cdot 360^\circ$, etc., we alternately get the values (1) and (2).

**Example 2.** Take the square root of 16. The argument of this number is $360^\circ k$ ($k$ an integer). The argument of the root is $360^\circ k : 2 = 180^\circ k$. If $k$ is zero or an even number, the argument of the root is equal to zero or is a multiple of 360°. Then $16^{\frac{1}{2}} = 4(\cos 0^\circ + i \sin 0^\circ) = 4$. But if $k$ is an odd number, then the argument will be 180° or will differ from 180° by a multiple of 360°. Then $16^{\frac{1}{2}} = 4(\cos 180^\circ + i \sin 180^\circ) = -4$.

**Example 3.** Extract the cube root of 1. The modulus of the root is $\sqrt[3]{1} = 1$. The argument of the radicand is $360^\circ k$ (where $k$ is any integer). The argument of the root is $120^\circ k$. Putting $k = 0, 1, 2$, we find three values of the argument of the root: 0°, 120°, 240°. The corresponding values of the root are *:

$$z_1 = \cos 0^\circ + i \sin 0^\circ = 1,$$

$$z_2 = \cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2} i,$$

$$z_3 = \cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$$

* It is useful to check these results. Multiplying the number $z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ by itself by the rule of Sec. 102, we find $z_2^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i = z_3$. Multiplying once again, we get $z_2^3 = z_3 z_2 = 1$. Verification of the root $z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$ is the same. Namely,

$$z_3^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i = z_2, \quad z_3^3 = z_2 z_3 = 1$$

In Fig. 17 these values are represented by the points $A_1$, $A_2$, $A_3$. The triangle $A_1A_2A_3$ is an equilateral triangle inscribed in a circle of radius 1.

**Example 4.** Take the sixth root of $-1$. The argument of the radicand $-1$ is $180 + 360^\circ k$. The argument of the

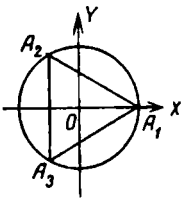



Fig. 17.

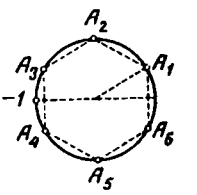



Fig. 18.

root is equal to $30^\circ + 60^\circ k$. We have the following six values of the root:

$$z_1 = \cos 30^\circ + i \sin 30^\circ = \frac{\sqrt{3}}{2} + \frac{1}{2} i,$$

$$z_2 = \cos 90^\circ + i \sin 90^\circ = i,$$

$$z_3 = \cos 150^\circ + i \sin 150^\circ = -\frac{\sqrt{3}}{2} + \frac{1}{2} i,$$

$$z_4 = \cos 210^\circ + i \sin 210^\circ = -\frac{\sqrt{3}}{2} - \frac{1}{2} i,$$

$$z_5 = \cos 270^\circ + i \sin 270^\circ = -i,$$

$$z_6 = \cos 330^\circ + i \sin 330^\circ = \frac{\sqrt{3}}{2} - \frac{1}{2} i$$

The points $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, $A_6$ which represent these values (Fig. 18) are the vertices of a regular hexagon.

From formula (B) it follows that the $n$ roots of some complex number and the $n$ roots of the conjugate of that number are pairwise conjugate.

**Example 5.** The fourth roots of the number $16(\cos 120^\circ + i \sin 120^\circ) = -8 + 8\sqrt{3}\, i$ are:

$$z_1 = 2(\cos 30^\circ + i \sin 30^\circ) = \sqrt{3} + i,$$

$$z_2 = 2(\cos 120^\circ + i \sin 120^\circ) = -1 + \sqrt{3} i,$$

$$z_3 = 2(\cos 210^\circ + i \sin 210^\circ) = -\sqrt{3} - i,$$

$$z_4 = 2(\cos 300^\circ + i \sin 300^\circ) = 1 - \sqrt{3} i$$

and the fourth roots of the number $16(\cos 120° - i \sin 120°) = -8 - 8\sqrt{3}i$ are

$$\bar{z}_1 = 2(\cos 30° - i \sin 30°) = \sqrt{3} - i,$$
$$\bar{z}_2 = 2(\cos 120° - i \sin 120°) = -1 - \sqrt{3}i,$$
$$\bar{z}_3 = 2(\cos 210° - i \sin 210°) = -\sqrt{3} + i,$$
$$\bar{z}_4 = 2(\cos 300° - i \sin 300°) = 1 + \sqrt{3}i$$

The numbers $z_1$ and $\bar{z}_1$, $z_2$ and $\bar{z}_2$, etc. are conjugate in pairs.

### 112. Raising a Complex Number to an Arbitrary Real Power

Raising a real number to a fractional power is defined in Sec. 125. However, only real values of the power are considered there. Here, we need a more general definition.

Let it be given by the following formula:

$$[r(\cos\varphi + i \sin\varphi)]^p = r^p(\cos p\varphi + i \sin p\varphi) \qquad (C)$$

Here, $p$ is any real number and $r^p$ denotes a positive number representing the $p$th power of the modulus $r$.

Formula (C) coincides with formula (A) of Sec. 110 when $p$ is integral and with (B) of Sec. 111 when $p$ is the fraction $\frac{1}{n}$. If $p$ is the fraction $\frac{m}{n}$, then by virtue of (C), (A) and (B)

$$[r(\cos\varphi + i \sin\varphi)]^{\frac{m}{n}} = \sqrt[n]{[r(\cos\varphi + i \sin\varphi)]^m} \qquad (D)$$

which is in agreement with the ordinary definition of a fractional power..

The fractional power of any complex (hence, also real) number has $n$ distinct values ($n$ is the denominator of the fraction). Formula (C) extends also to any irrational exponent $p$, in which case the $p$th power of any number has an infinite number of values.

**Example 1.** Raise the number $-16$ to the power $\frac{3}{4}$.

We have

$$p = \frac{3}{4}, \quad r = 16, \quad \varphi = 180° + 360°k$$

The modulus of the power $(-16)^{\frac{3}{4}}$ is, by (C), $16^{\frac{3}{4}}=8$. The argument of the power is equal to

$$\frac{3}{4}(180°+360°k)=135°+270°k$$

Assuming $k=0, 1, 2, 3$ (the other integral values of $k$ will not yield fresh results), we have the following four values of the power:

$$z_1=8(\cos 135°+i\sin 135°)=-4\sqrt{2}+4\sqrt{2}i,$$
$$z_2=8[\cos(135°+270°)+i\sin(135°+270°)]$$
$$=8(\cos 45°+i\sin 45°)=4\sqrt{2}+4\sqrt{2}i,$$
$$z_3=8[\cos(135°+2\cdot 270°)+i\sin(135°+2\cdot 270°)]$$
$$=8[\cos(-45°)+i\sin(-45°)]=4\sqrt{2}-4\sqrt{2}i,$$
$$z_4=8[\cos(135°+3\cdot 270°)+i\sin(135°+3\cdot 270°)]$$
$$=8[\cos(-135°)+i\sin(-135°)]=-4\sqrt{2}-4\sqrt{2}i$$

These values are represented by the points $B_1$, $B_2$, $B_3$, $B_4$ (Fig. 19).

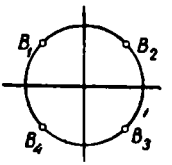



Fig. 19.

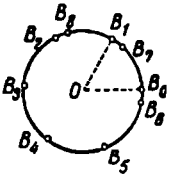



Fig. 20.

**Example 2.** Raise the number 1 to the power $\frac{1}{2\pi}$. Here, $p=\frac{1}{2\pi}$, $r=1$, $\varphi=360°k$. By (C) we have

$$1^{\frac{1}{2\pi}}=\cos\frac{360°}{2\pi}k+i\sin\frac{360°}{2\pi}k$$

Figure 20 shows the points $B_0$, $B_1$, $B_2$, $B_3$, ... depicting the values of the power which result when $k=0, 1, 2, 3$. All lie on a circle of radius 1. No pairs of points coincide. Indeed, each of the angles $B_0OB_1$, $B_1OB_2$, etc. is equal to a radian, i.e., each of the arcs $B_0B_1$, $B_1B_2$, etc. is one

radius in length. If some point $B_1$ coincided with $B_0$, this would mean that the circle traversed $s$ times ($s$ a whole number) contained $l$ radii. But then a single circuit of the circle would be of length exactly equal to $\frac{l}{s}$ radii, yet the circumference of a circle is not commensurable with its radius. Hence, no pair of points $B_0$, $B_1$, ... can be coincident. The more points we take, the more densely they cover the circle. An infinity of points $B$ accumulate about any point of the circle circumference. And yet on the circumference there are everywhere sites not occupied by any points $B$. Such, for instance, is the point which is diametrically opposite $B_0$, or any vertex of any regular polygon in which $B_0$ is one of the vertices.

**Note.** It is also possible to define the power of a complex number for a complex exponent. It too has an infinity of values, but the corresponding points do not, in the general case, accumulate. They are spread out.

### 113. Some Facts about Higher Degree Equations

For general-form equations of third and fourth degree (see Sec. 66), we have formulas which express the roots of the equation in terms of the literal values of the coefficients. These formulas involve radicals of index 2 and 3. They are complicated and too unwieldy for practical use. No such formulas exist for higher-degree equations. It has been proved that it is impossible to express the roots of a general equation of degree higher than fourth in terms of literal coefficients by means of a finite number of additions, subtractions, multiplications, divisions, involutions and evolutions. This is possible only for certain particular types of literal equations of higher degree.

Nevertheless, the roots of any algebraic equation with numerical coefficients can be found in approximate fashion to any desired degree of accuracy.

Prior to the introduction of complex numbers, even a quadratic equation did not always have a solution (see Sec. 92). With the advent of complex numbers, every algebraic equation has at least one root (the coefficients of the algebraic equation may be quite arbitrary, even complex).

An equation of the $n$th degree cannot have more than $n$ distinct roots, though it may have a smaller number. For instance, the quintic (fifth-degree) equation $(x-3)(x-2)$

$\times(x-1)^3=0$, which in expanded form looks like this: $x^6-8x^4+24x^3-34x^2+23x-6=0$, has the roots $x_1=3$, $x_2=2$, $x_3=1$, and no other roots. Still, it is considered that this equation has five roots: $x_1=3$, $x_2=2$, $x_3=1$, $x_4=1$, $x_5=1$. The root 1 is counted three times because the left member of the equation involves the factor $x-1$ to the third power.

Counting in this manner, we find that every $n$th degree equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0 \qquad (a_0\neq 0) \tag{1}$$

has exactly $n$ roots. The reason is this. Equation (1) may be represented (uniquely) as

$$a_0(x-x_1)(x-x_2)\ \ldots\ (x-x_n)=0 \tag{2}$$

The numbers $x_1, x_2, \ldots, x_n$ are the roots of (1). There may be, among them, several with the same value (in the last example, we had $x_3=x_4=x_5=1$). This value is counted as a root as many times as it is repeated. If counted in this fashion, the total number of roots is always equal to $n$.

If the coefficients of an algebraic equation are real and one of the roots is a complex number $a+bi$, then the conjugate complex number $a-bi$ is also a root. For instance, the complex number $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ is a root of the equation $x^4+1=0$ (Sec. 111); the conjugate complex number $\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$ is also a root of this equation. Thus, an equation with real coefficients always has an even number of complex roots.

Every odd-degree equation with real coefficients has at least one real root (there is always an even number of complex roots and the total number of roots, by hypothesis, is odd).

The sum of the roots of equation (1) is $-\frac{a_1}{a_0}$, while the product of the roots is equal to $(-1)^n\frac{a_n}{a_0}$. These properties were pointed out by the French mathematician Viète in 1591. Viète did not recognize negative numbers (cf. Sec. 67) and so he considered the case when all roots are positive.

**Example.** The equation $x^5-8x^4+24x^3-34x^2+23x-6=0$ ($n=5$, $a_0=1$, $a_1=-8$, $a_n=-6$) has the following roots (see above): 3, 2, 1, 1, 1. Their sum is 8 $\left(\text{i. e.}-\frac{-8}{1}\right)$ and their product 6 $\left[\text{i. e. }(-1)^5\cdot\frac{-6}{1}\right]$.

These properties (and other similar properties) are derived from a comparison of equations (1) and (2) (all terms in them must be the same, in particular the second and the last).

### 114. Fundamentals of Inequalities

Two expressions, numerical or literal, connected by the sign $>$ (greater than) or the sign $<$ (less than) form an *inequality* (numerical or literal).

Every true numerical inequality and every literal inequality that is valid for all numerical real values of the letters involved is called an *absolute* or *unconditional inequality*.

**Example 1.** The numerical inequality $2 \cdot 3 - 5 < 8 - 5$ (it is true!) is an unconditional inequality.

**Example 2.** The literal inequality $a^2 > -2$ is unconditional since for any numerical (real) value of $a$ the quantity $a^2$ is positive or equal to zero and, hence, is always greater than $-2$.

Two expressions can also be connected by the signs $\leqslant$ (less than or equal to) and $\geqslant$ (greater than or equal to). Thus, the notation $2a \geqslant 3b$ means that the quantity $2a$ is either greater than $3b$ or equal to $3b$. These expressions are also called inequalities.

The literal quantities involved in an inequality can be classified as known and unknown. It is common practice to stipulate which letters are to be taken for the unknown quantities and the known quantities. Ordinarily, the last letters of the alphabet ($x$, $y$, $z$, $u$, $v$ etc.) are used for unknowns.

To solve an inequality means to indicate the limits within which the real values of the unknown quantities must lie in order for the inequality to be true.

If several inequalities are given, then to solve the system of inequalities means to indicate the limits within which the values of the unknowns must lie so that the given inequalities are true.

**Example 3.** Solve the inequality $x^2 < 4$. This inequality is true if $|x| < 2$, that is, if $x$ lies between $-2$ and $+2$. The solution is of the form $-2 < x < 2$.

**Example 4.** Solve the inequality $2x > 8$.

The solution looks like this: $x > 4$. Here $x$ is bounded on one side only.

**Example 5.** The inequality $(x-2)(x-3) > 0$ is true if $x > 3$ [then both factors $(x-2)$, $(x-3)$ are positive] and also if $x < 2$ (then both factors are negative) and is not true when $x$ lies between 2 and 3 (and also when $x=2$ and $x=3$). Therefore, the solution is given by two inequalities:

$$x > 3,\ x < 2$$

**Example 6.** The inequality $x^2 < -2$ has no solution (cf. Example 2).

### 115. Basic Properties of Inequalities

1. If $a > b$, then $b < a$; conversely, if $a < b$, then $b > a$.

**Example 1.** If $5x-1 > 2x+1$, then $2x+1 < 5x-1$.

2. If $a > b$ and $b > c$, then $a > c$. In the same way, if $a < b$ and $b < c$, then $a < c$.

**Example 2.** From the inequalities $x > 2y$, $2y > 10$ it follows that $x > 10$.

3. If $a > b$, then $a+c > b+c$ (and $a-c > b-c$), and if $a < b$, then $a+c < b+c$ (and $a-c < b-c$), which means that *we can add to (or subtract from) both sides of an inequality the same quantity* without changing the sense of the inequality.

**Example 3.** Given the inequality $x+8 > 3$. Subtracting 8 from both sides we get $x > -5$.

**Example 4.** Given the inequality $x-6 < -2$. Adding 6 to both sides, we have $x < 4$.

4. If $a > b$ and $c > d$, then $a+c > b+d$; in the same way, if $a < b$ and $c < d$, then $a+c < b+d$; that is, *two inequalities having the same sense* (the expression "inequalities of the same sense" means that both inequalities involve the sign $>$ or both have the sign $<$) *may be combined term by term*. This holds true for any number of inequalities, say, if $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$, then $a_1+a_2+a_3 > b_1+b_2+b_3$.

**Example 5.** The inequalities $-8 > -10$ and $5 > 2$ are true. Combining them termwise, we find the true inequality $-3 > -8$.

**Example 6.** Given the system of inequalities $\frac{1}{2}x+\frac{1}{2}y < 18$, $\frac{1}{2}x-\frac{1}{2}y < 4$. Adding them term by term, we get $x < 22$.

**Note.** Two inequalities of the same sense cannot be subtracted termwise one from the other because the result may be either true or untrue. For example if from the inequality $10 > 8$

we subtract termwise the inequality $2 > 1$, we have the true inequality $8 > 7$, but if we subtract the inequality $6 > 1$ from that inequality, we get an absurd result. Compare with the following item.

5. If $a > b$ and $c < d$, then $a - c > b - d$; if $a < b$ and $c > d$, then $a - c < b - d$; that is, from one inequality it is possible to subtract termwise another inequality of opposite sense* and leave unchanged the sense of the inequality from which the other was subtracted.

**Example 7.** The inequalities $12 < 20$ and $15 > 7$ are true. Subtracting the first from the second term by term and retaining the sign of the first, we get the true inequality $-3 < 13$. Subtracting the first from the second and leaving the sign of the second, we obtain the true inequality $3 > -13$.

**Example 8.** Given a system of inequalities: $\frac{1}{2}x + \frac{1}{2}y < 18$, $\frac{1}{2}x - \frac{1}{2}y > 8$. Subtracting the second one from the first, we get $y < 10$.

6. If $a > b$ and $m$ is a positive number, then $ma > mb$ and $\frac{a}{m} > \frac{b}{m}$, or

*both sides of an inequality may be divided or multiplied by the same positive number without changing the sense of the inequality.*

But if $a > b$ and $n$ is a negative number, then $na < nb$ and $\frac{a}{n} < \frac{b}{n}$, or

*both sides of an inequality may be multiplied or divided by the same negative number, but then the sense of the inequality is reversed* (it is of course forbidden to multiply and divide both members of an inequality by zero).

**Example 9.** Dividing both sides of the true inequality $25 > 20$ by 5, we get the true inequality $5 > 4$. But if we divide both sides of $25 > 20$ by $-5$, we have to reverse the sign $>$ (to $<$) to get the true inequality $-5 < -4$.

**Example 10.** From the inequality $2x < 12$ it follows that $x < 6$.

**Example 11.** From the inequality $-\frac{1}{3}x > 4$ it follows that $x < -12$.

**Example 12.** From the inequality $\frac{x}{k} > \frac{y}{l}$ it follows that $lx > ky$ if the signs of the numbers $l$ and $k$ are the same, and that $lx < ky$ if the signs of the numbers $l$ and $k$ are different.

* The expression "inequalities of opposite sense" means that one of the inequalities has the sign $>$ and the other, the sign $<$.

### 116. Some important inequalities

1. $|a+b| \leqslant |a|+|b|$. Here, $a$ and $b$ are arbitrary real or complex numbers (but $|a|$, $|b|$ and $|a+b|$ are always real and positive, see Secs. 69 and 105), that is, *the modulus of a sum does not exceed the sum of the moduli.* Equality occurs only when both numbers $a$ and $b$ have the same argument (Sec. 105), in particular when both numbers are positive or both negative.

**Example 1.** Let $a=+3$, $b=-5$. Then $a+b=-2$, $|a+b|=2$, $|a|=3$ $|b|=5$. We have $2<3+5$.

**Example 2.** Let $a=4+3i$, $b=6-8i$. Then

$$a+b=10-5i,\ |a+b|=\sqrt{10^2+(-5)^2}=\sqrt{125},$$
$$|a|=\sqrt{4^2+3^2}=5,\ |b|=\sqrt{6^2+(-8)^2}=10,$$
$$|a|+|b|=15$$

We have $\sqrt{125}<15$.

**Note.** The inequality $|a+b| \leqslant |a|+|b|$ may be extended to a greater number of terms; thus

$$|a+b+c| \leqslant |a|+|b|+|c|$$

2. $a+\frac{1}{a} \geqslant 2$ ($a$ is a positive number). The equality holds only when $a=1$.

3. $\sqrt{ab} \leqslant \frac{a+b}{2}$ ($a$ and $b$ positive numbers); what this means is that *the geometric mean* (Sec. 59) *of two numbers does not exceed their arithmetic mean.* The equality $\sqrt{ab}=\frac{a+b}{2}$ holds only when $a=b$.

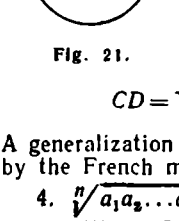


Fig. 21.

**Example 3.** $a=2$, $b=8$, $\sqrt{ab}=4$, $\frac{a+b}{2}=5$, we have $4<5$. This inequality was known 2000 years ago. Its obvious nature is seen geometrically in Fig. 21, where

$$CD=\sqrt{AD\cdot DB} \text{ and } CO=AO=\frac{AD+DB}{2}$$

A generalization of it is the following inequality established by the French mathematician Cauchy in 1821.

4. $\sqrt[n]{a_1a_2\ldots a_n} \leqslant \frac{a_1+a_2+\ldots+a_n}{n}$ (the numbers $a_1, a_2, \ldots, a_n$ are positive). The equality is valid only when all numbers $a_1, a_2, \ldots, a_n$ are equal.

5. $1:\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) \leqslant \sqrt{ab}$ ($a$ and $b$ positive). The sign of equality is valid only when $a=b$.

**Example 4.** $a=2$. $b=8$, $1:\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{16}{5}$; we have $\frac{16}{5}<4$.

The quantity $1:\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{2ab}{a+b}$ is a mean quantity (Sec. 59) between $a$ and $b$. It is called the *harmonic mean*.* Thus, the harmonic mean between two quantities does not exceed the arithmetic mean of the quantities. This property can be generalized to any number of quantities; in conjunction with the inequality of Item 4 we have

$$1:\frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) \leqslant \sqrt[n]{a_1a_2\ldots a_n} \leqslant \frac{a_1+a_2+\ldots+a_n}{n}$$

6. $$\left|\frac{a_1+a_2+\ldots+a_n}{n}\right| \leqslant \sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}$$

(the numbers $a_1$, $a_2$, ..., $a_n$ are arbitrary); thus, the absolute value of the arithmetic mean does not exceed the root-mean-square (Sec. 61). The equals sign holds only when $a_1=a_2=\ldots=a_n$.

**Example 5.** $a_1=3$, $a_2=4$, $a_3=5$, $a_4=6$.

Here the arithmetic mean is $\frac{a_1+a_2+a_3+a_4}{4}=\frac{9}{2}$ and the root-mean-square is

$$\sqrt{\frac{a_1^2+a_2^2+a_3^2+a_4^2}{4}}=\sqrt{\frac{9+16+25+36}{4}}=\frac{\sqrt{86}}{2}$$

We have $\frac{9}{2}<\frac{\sqrt{86}}{2}$.

7. $$a_1b_1+a_2b_2+\ldots+a_nb_n \leqslant \sqrt{a_1^2+a_2^2+\ldots+a_n^2}\times \sqrt{b_1^2+b_2^2+\ldots+b_n^2}$$

the numbers $a_1$, $a_2$, ..., $a_n$; $b_1$, $b_2$, ..., $b_n$ are arbitrary. The equality is valid only when $a_1:b_1=a_2:b_2=\ldots=a_n:b_n$.

**Example 6.** Let $a_1=1$, $a_2=2$, $a_3=5$; $b_1=-3$, $b_2=1$, $b_3=2$. We have $a_1b_1+a_2b_2+\ldots+a_nb_n=1\cdot(-3)+2\cdot1$

* In the ancient Greek theory of musical harmony an important role was played by the harmonic mean between the lengths of two strings. Whence the name "harmonic".

$+5\cdot 2=9;$

$$\sqrt{a_1^2+a_2^2+\ldots+a_n^2}=\sqrt{1^2+2^2+5^2}=\sqrt{30},$$
$$\sqrt{b_1^2+b_2^2+\ldots+b_n^2}=\sqrt{(-3)^2+1^2+2^2}=\sqrt{14}$$

We have $9<\sqrt{30}\cdot\sqrt{14}$.

**8. Chebyshev inequalities.** Let the numbers $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be positive.

If $a_1\leqslant a_2\leqslant\ldots\leqslant a_n$ and $b_1\leqslant b_2\leqslant\ldots\leqslant b_n$, then

$$\frac{a_1+a_2+\ldots+a_n}{n}\cdot\frac{b_1+b_2+\ldots+b_n}{n}\leqslant\frac{a_1b_1+a_2b_2+\ldots+a_nb_n}{n} \quad (1)$$

But if $a_1\leqslant a_2\leqslant\ldots\leqslant a_n$ yet $b_1\geqslant b_2\geqslant\ldots\geqslant b_n$, then

$$\frac{a_1+a_2+\ldots+a_n}{n}\cdot\frac{b_1+b_2+\ldots+b_n}{n}\geqslant\frac{a_1b_1+a_2b_2+\ldots+a_nb_n}{n} \quad (2)$$

In both cases, the equality is valid only when all the numbers $a_1, a_2, \ldots, a_n$ are equal and also the numbers $b_1, b_2, \ldots, b_n$ are equal.

**Example 7.** Let $a_1=1$, $a_2=2$, $a_3=7$ and $b_1=2$, $b_2=3$, $b_3=4$. Then

$$\frac{a_1+a_2+\ldots+a_n}{n}=\frac{1+2+7}{3}=\frac{10}{3},$$
$$\frac{b_1+b_2+\ldots+b_n}{n}=\frac{2+3+4}{3}=3,$$
$$\frac{a_1b_1+a_2b_2+\ldots+a_nb_n}{n}=\frac{1\cdot 2+2\cdot 3+7\cdot 4}{3}=12$$

We have

$$\frac{10}{3}\cdot 3<12$$

**Example 8.** Let $a_1=1$, $a_2=2$, $a_3=7$ and $b_1=4$, $b_2=3$, $b_3=2$. Then

$$\frac{a_1+a_2+a_3}{3}=\frac{10}{3},\quad\frac{b_1+b_2+b_3}{3}=3,$$
$$\frac{a_1b_1+a_2b_2+a_3b_3}{3}=8$$

We have

$$\frac{10}{3}\cdot 3>8$$

The inequalities (1) and (2) may be stated thus:

If two sequences of positive quantities have the same number of terms and if in both sequences the terms do not diminish (or do not increase), then the product of the arithmetic means does not exceed the arithmetic mean of the pro-

ducts. But if in one sequence the terms do not decrease and in the other they do not increase, then the opposite inequality is valid.

These inequalities were discovered in 1886 by the celebrated Russian mathematician P. L. Chebyshev (1821—1894). He also generalized them and proved the following inequalities:

If $0 < a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ and $0 < b_1 \leqslant b_2 \leqslant \ldots \leqslant b_n$,

then

$$\sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}\sqrt{\frac{b_1^2+b_2^2+\ldots+b_n^2}{n}} \leqslant \sqrt{\frac{(a_1b_1)^2+(a_2b_2)^2+\ldots+(a_nb_n)^2}{n}}, \tag{3}$$

$$\sqrt[3]{\frac{a_1^3+a_2^3+\ldots+a_n^3}{n}}\sqrt[3]{\frac{b_1^3+b_2^3+\ldots+b_n^3}{n}} \leqslant \sqrt[3]{\frac{(a_1b_1)^3+(a_2b_2)^3+\ldots+(a_nb_n)^3}{n}} \tag{4}$$

and so on.

However if $0 < a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ but $b_1 \geqslant b_2 \ldots \geqslant b_n > 0$, then the opposite inequalities hold true.

### 117. Equivalent Inequalities. Basic Techniques for Solving Inequalities

Two inequalities involving the same unknowns are called *equivalent* if they are true for the same values of the unknowns.

The equivalence of two systems of inequalities is defined in the same manner.

**Example 1.** The inequalities $3x+1>2x+4$ and $3x>2x+3$ are equivalent since both are true for $x>3$ and both are untrue when $x \leqslant 3$.

**Example 2.** The inequalities $2x \leqslant 6$ and $x^2 \leqslant 9$ are not equivalent since the solution of the former is $x \leqslant 3$, while the solution of the latter is $-3 \leqslant x \leqslant 3$, so that, say for $x=-4$, the former is true and the latter untrue.

The process of solving an inequality consists, in the main, in replacing the given inequality (or the given system of inequalities) by other, equivalent, inequalities (see Sec. 215 for a graphical solution of inequalities). When solving inequalities use the following basic techniques (cf. Sec. 82).

1. Replacing one expression by an identical expression.

2. Transposing a term from one side of the inequality to the other with a reversal of sign (by virtue of Sec. 115, Item 3).

3. Multiplying or dividing both sides of an inequality by the same numerical quantity (not equal to zero). If the multiplier is positive, then the sense of the inequality remains unchanged, if it is negative, the sense of the inequality is reversed (Sec. 115, Item 6).

Each one of these transformations yields an inequality that is equivalent to the original one.

**Example 3.** Given the inequality $(2x-3)^2 < 4x^2+2$. Replace the left member by the identical expression $4x^2-12x+9$. We get $4x^2-12x+9<4x^2+2$. Transpose $4x^2$ to the left member and 9 to the right member. Collecting like terms, we get $-12x<-7$. Divide both sides of the inequality by $-12$; this necessitates reversing the sense of the inequality. The solution of the given inequality is $x > \frac{7}{12}$.

To multiply (and of course divide) an inequality by zero is not permissible. When multiplying or dividing both members of an inequality by literal expressions, we get an inequality which, as a rule, is not equivalent to the original one.

**Example 4.** Given the inequality $(x-2)\,x < x-2$.

Divide both sides by $x-2$ to get $x < 1$. But this inequality is not equivalent to the original one because, for instance, the value $x=0$ does not satisfy the inequality $(x-2)\,x < x-2$. Again, the inequality $x > 1$ is not equivalent to the original one because, for example, the value $x=3$ does not satisfy the inequality $(x-2)\,x < x-2$.

### 118. Inequalities Classified

Inequalities involving unknown quantities may be divided into *algebraic* and *transcendental inequalities*. Algebraic inequalities are further subdivided into *inequalities of first, second*, etc. *degree*. This classification is the same as that for equations (Sec. 83).

**Example 1.** The inequality $3x^2-2x+5>0$ is an algebraic inequality of the second degree.

**Example 2.** The inequality $2^x > x+4$ is a transcendental inequality.

**Example 3.** The inequality $3x^2-2x+5>3x\,(x-2)$ is an algebraic inequality of the first degree because it can be reduced to the inequality $4x+5>0$.

### 119. Inequalities of the First Degree in One Unknown

A first-degree (linear) inequality in one unknown can be reduced to the form

$$ax > b$$

It has the solution

$$x > \frac{b}{a} \quad \text{if} \quad a > 0$$

and

$$x < \frac{b}{a} \quad \text{if} \quad a < 0$$

**Example 1.** Solve the inequality $5x-3>8x+1$.

**Solution.** $5x-8x>3+1$, $-3x>4$, $x<-\frac{4}{3}$.

**Example 2.** Solve the inequality $5x+2<7x+6$.

**Solution.** $5x-7x<6-2$, $-2x<4$, $x>-2$.

**Example 3.** Solve the inequality $(x-1)^2<x^2+8$.

**Solution.** $x^2-2x+1<x^2+8$, $-2x<7$, $x>-\frac{7}{2}$.

**Note.** An inequality of the form $ax+b>a_1x+b_1$ is an inequality of the first degree if $a$ and $a_1$ are distinct. Otherwise the inequality is reduced to a numerical (true or untrue) inequality.

**Example 4.** Given the inequality $2(3x-5)<3(2x-1)+5$. It is equivalent to the inequality $6x-10<6x+2$ and the latter can be reduced to the numerical (unconditional) inequality $-10<2$. Hence, the original inequality is unconditional.

**Example 5.** The inequality $2(3x-5)>3(2x-1)+5$ reduces to the meaningless numerical inequality $-10>2$. Hence, the original inequality has no solutions.

### 120. A System of Inequalities of the First Degree

To solve a system of first-degree inequalities, find the solution of each inequality separately and compare the solutions. This comparison either yields the solution of the system or reveals that the system does not have any solutions.

**Example 1.** Solve the system of inequalities

$$4x-3>5x-\ ,\quad 2x+4<8x$$

The solution of the first inequality is $x < 2$; of the second, $x > \frac{2}{3}$. The solution of the system is $\frac{2}{3} < x < 2$.

**Example 2.** Solve the system of inequalities

$$2x-3 > 3x-5, \quad 2x+4 > 8x$$

For the first we have the solution $x < 2$; for the second, $x < \frac{2}{3}$. The solution of the system is $x < \frac{2}{3}$ (under this condition the inequality $x < 2$ is certainly true).

**Example 3.** Solve the system of inequalities

$$2x-3 < 3x-5, \quad 2x+4 > 8x$$

The solution of the first inequality is $x > 2$, the solution of the second, $x < \frac{2}{3}$. These conditions are contradictory. The system has no solutions.

**Example 4.** Solve the system of inequalities

$$2x < 16, \quad 3x+1 > 4x-4, \quad 3x+6 > 2x+7, \quad x+5 < 2x+6$$

The solutions are, respectively, $x < 8$, $x < 5$, $x > 1$, $x > -1$. Comparing these conditions we find that the first two may be replaced by the second alone, and the third and fourth, by the third alone. The solution of the system is $1 < x < 5$.

### 121. Elementary Inequalities of the Second Degree in One Unknown

1. The inequality $x^2 < m$. (1)

(a) If $m > 0$, the solution is

$$-\sqrt{m} < x < \sqrt{m} \tag{1a}$$

(b) If $m \leqslant 0$, then there is no solution (the square of a real number cannot be negative)

2. The inequality $x^2 > m$. (2)

a) If $m > 0$, then (2) is valid, firstly, for all values of $x$ greater than $\sqrt{m}$, and, secondly, for all values of $x$ less than $-\sqrt{m}$:

$$x > \sqrt{m} \quad \text{or} \quad x < -\sqrt{m} \tag{2a}$$

(b) If $m = 0$, then (2) holds true for all $x$ except $x = 0$:

$$x > 0 \quad \text{or} \quad x < 0 \tag{2b}$$

(c) If $m < 0$, then (2) is an unconditional inequality.

**Example 1.** The inequality $x^2 < 9$ has the solution $-3 < x < 3$.

**Example 2.** The inequality $x^2 < -9$ has no solutions.

**Example 3.** The inequality $x^2 > 9$ has for a solution the set of all numbers greater than 3 and the set of all numbers less than $-3$.

**Example 4.** The inequality $x^2 > -9$ is unconditional.

### 122. Inequalities of the Second Degree in One Unknown (General Case)

Dividing a second-degree inequality by the coefficient of $x^2$, we reduce it to one of the following types:

$$x^2+px+q < 0, \tag{1}$$

$$x^2+px+q > 0 \tag{2}$$

Transpose the constant term to the right side and add to both sides $\left(\frac{p}{2}\right)^2$. This yields, respectively,

$$\left(x+\frac{p}{2}\right)^2 < \left(\frac{p}{2}\right)^2-q, \tag{1'}$$

$$\left(x+\frac{p}{2}\right)^2 > \left(\frac{p}{2}\right)^2-q \tag{2'}$$

If we denote $x+\frac{p}{2}$ by $z$ and $\left(\frac{p}{2}\right)^2-q$ by $m$, we get the elementary inequalities

$$z^2 < m, \tag{1''}$$

$$z^2 > m \tag{2''}$$

The solution of these inequalities was given in the preceding section. Knowing it, we find the solution of (1) or (2).

**Example 1.** Solve the inequality $-2x^2+14x-20 > 0$. Divide both members by $-2$ (Sec. 117, Item 3) to get $x^2-7x+10 < 0$. Transposing the constant term 10 to the right and adding $\left(\frac{7}{2}\right)^2$ to both members, we get $\left(x-\frac{7}{2}\right)^2 < \frac{9}{4}$, whence (Sec. 121, Case 1a)

$$-\frac{3}{2} < x-\frac{7}{2} < \frac{3}{2}$$

Adding $\frac{7}{2}$, we find $-\frac{3}{2}+\frac{7}{2} < x < \frac{3}{2}+\frac{7}{2}$, that is, $2 < x < 5$.

**Example 2.** Solve the inequality $-2x^2+14x-20<0$. Performing the same transformations, we get the inequality $\left(x-\frac{7}{2}\right)^2>\frac{9}{4}$, whence (Sec. 121, Case 2a) we find that our inequality is valid, firstly, for $x-\frac{7}{2}>\frac{3}{2}$, that is, for $x>5$, and, secondly, for $x-\frac{7}{2}<-\frac{3}{2}$, or for $x<2$.

**Example 3.** Solve the inequality $x^2+6x+15<0$. Transposing the constant term to the right and adding to both members $\left(\frac{6}{2}\right)^2$, i.e., 9, we find $(x+3)^2<-6$. This inequality (Sec. 121, Case 1b) has no solutions and so the given inequality has no solutions.

**Example 4.** Solve the inequality $x^2+6x+15>0$. As in Example 3, we find $(x+3)^2>-6$. This inequality (Sec. 121, Case 2c) is unconditional.

### 123. Arithmetic Progressions

At one time the term progression was used in mathematics to denote any sequence of numbers generated by some law that permitted extending the sequence indefinitely in one direction. For example, by squaring the sequence of whole numbers, we get the sequence 1, 4, 9, 16, 25, etc. This can be continued without end simply by applying the law of formation of the sequence. The numbers thus generated are called the *terms* of the sequence. At the present time the term "progression" is largely confined to the two most important kinds of number sequences: arithmetic and geometric, all other progressions of numbers being termed *sequences.*

An arithmetic progression is a sequence of numbers such that the difference between any two successive terms is a constant called the *common difference.*

**Example 1.** The sequence of natural numbers 1, 2, 3, 4, 5, ... is an arithmetic progression with common difference 1.

**Example 2.** The sequence of numbers 10, 8, 6, 4, 2, 0, $-2$, $-4$ ... is an arithmetic progression with common difference $-2$.

*Any term of an arithmetic progression* may be computed by the formula

$$a_n=a_1+d(n-1)$$

where $a_1$ is the first term of the progression, $d$ is the com-

mon difference, and $a_n$ is the $n$th term, $n$ being the number of the term.

*The sum of the first n terms of an arithmetic progression* is given by the formula

$$s_n = \frac{(a_1 + a_n) \cdot n}{2}$$

**Example 3.** In the progression 12, 15, 18, 21, 24, ... the tenth term is egual to $a_{10} = 12 + 3 \cdot 9 = 39$.

The sum of the first ten terms is

$$s_{10} = \frac{(a_1 + a_{10})\, 10}{2} = \frac{(12 + 39)\, 10}{2} = 255$$

**Example 4.** The sum of all integers from 1 to 100 inclusive is $\frac{(1+100)\, 100}{2} = 5050$.

### 124. Geometric Progressions

A geometric progression is a sequence of numbers such that the ratio of each term to the immediately preceding one is a constant called the *common ratio* (or, simply, *ratio*).

**Example 1.** The numbers 5, 10, 20, 40, ... form a geometric progression with ratio 2.

**Example 2.** The numbers 1, 0.1, 0.01, 0.001, etc. constitute a geometric progression with ratio 0.1.

A geometric progression is termed *increasing* when the absolute value of its ratio is greater than unity (as in Example 1) and *decreasing* when it is less than unity (as in Example 2).

**Note.** The ratio of a progression may be a negative number, but such progressions are of no practical significance.

*Any term of a geometric progression* may be computed from the formula

$$a_n = a_1 q^{n-1} \tag{1}$$

where $a_1$ is the first term, $q$ is the ratio, and $a_n$ is the $n$th term, $n$ being the number of the term.

*The sum of the first n terms of a geometric progression* (whose ratio is not unity) is given by the formula

$$s_n = \frac{a_n q - a_1}{q - 1} = \frac{a_1 - a_n q}{1 - q} \tag{2}$$

The first expression is most conveniently used for increasing progressions, the second for decreasing progressions.

But if $q=1$, then the progression consists of equal terms and in place of (2) we have $s_n=na_1$.

**Example 3.** In the geometric progression, 5, 10, 20, 40, ... the tenth term $a_{10}=5\cdot 2^9=5\cdot 512=2560$. The sum of the first ten terms is

$$s_{10}=\frac{a_{10}\cdot 2-a_1}{2-1}=5115$$

*The sum of an infinitely decreasing progression* is a number approached without bound by the sum of the first $n$ terms of the decreasing progression when the number $n$ increases without bound.

The sum of an infinitely decreasing progression is given by the formula

$$s=\frac{a_1}{1-q}$$

**Example 4.** The sum of the infinite geometric progression $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \left(a_1=\frac{1}{2}, q=\frac{1}{2}\right)$ is $\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$ that is, the sum $\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}$ approaches the number 1 without bound as $n$ increases without bound.

### 125. Negative, Zero and Fractional Exponents

Raising a quantity to the $n$th power was originally understood as an $n$-fold repetition of a certain number as a factor. From this point of view, such expressions as $9^{-2}$ or $9^{1\frac{1}{2}}$ appear to be meaningless since it is clearly impossible to take 9 as a factor for a total of minus two times or one and one half times. Nevertheless, mathematicians attach a very definite meaning to such expressions; namely, $9^{-2}$ is considered to be equal to $\frac{1}{9^2}=\frac{1}{81}$; $9^{1\frac{1}{2}}$, to $\sqrt{9^3}=(\sqrt{9})^3=27$, etc. Here again we encounter the generalization of the concept of a mathematical operation that is constantly going on in mathematics. The simplest and earliest generalization of this kind was that of the operation of multiplication to the case of a fractional factor (see Sec. 34). It is possible to get along without introducing

fractional or negative exponents, but then problem of a single kind would have to be solved by a multitude of diverse rules instead of one. The problems we have in mind are to be found mostly in higher mathematics and therefore many concrete examples are beyond the scope of this book. However, one of these problems is studied in detail in elementary mathematics. It has to do with logarithms (see Sec. 126). It is worth noting that the theory of logarithms, which today is intimately bound up with the generalization of the concept of a power, dispensed with fractional and negative exponents for a whole century after its discovery (at the turn of the 17th century); the same goes for the problems of higher mathematics that we mentioned. It was only at the end of the 17th century that the number and complexity of mathematical problems urgently called for a generalization of the concept of a power. That was the path taken by certain scholars and, most notably, Newton.

**Definition of negative power.*** By definition, the power of any number with a negative (integral) exponent is unity divided by the power of that number with a positive exponent, the value of which is equal to the absolute value of the negative exponent; or

$$a^{-m}=\frac{1}{a^m}$$

**Examples.** $2^{-3}=\frac{1}{2^3}=\frac{1}{8}$, $\left(\frac{3}{4}\right)^{-2}=1:\left(\frac{3}{4}\right)^2=\frac{16}{9}$,

$$(-4)^{-3}=1:(-4)^3=-\frac{1}{64}, \text{ etc.}$$

The equation $a^{-m}=1:a^m$ holds true both for positive $m$ and for negative $m$. If, say, $m=-5$, then $-m$ will be equal to $+5$ and our formula will look like this: $a^5=\frac{1}{a^{-5}}$, which is in agreement with the definition given above.

Operations involving negative exponents obey all the rules that hold for positive exponents. What is more, it was only after the introduction of negative exponents that the rules for handling positive exponents acquired full generality.

Thus, the formula $a^m:a^n=a^{m-n}$ (see Sec. 89) can now be applied not only when $m>n$, but also when $m<n$.

**Example.** $a^5:a^8=a^{5-8}=a^{-3}$. Indeed, according to the definition $a^{-3}=\frac{1}{a^3}$ so that the expression $a^5:a^8=a^{-3}$ means

* The terms "negative power", "zeroth power", and "fractional power" are taken to mean powers with a negative, zero and fractional exponent, respectively.

$\frac{a^5}{a^5}=\frac{1}{a^3}$. For the formula $a^m:a^n=a^{m-n}$ to have generality, it must hold true when $m=n$ as well. For this purpose we make the following definition.

**Definition of a zeroth power.** *The zeroth power of any nonzero number is unity* [the expression $0^0$, like the expression $\frac{0}{0}$ (see Sec. 37), is indeterminate].

**Examples.** $3^0=1$, $(-3)^0=1$, $\left(-\frac{2}{3}\right)^0=1$, $a^5:a^5=a^0=1$.

**Definition of a fractional power.** *To raise a* (real) *number a to the power* $\frac{m}{n}$ *means to extract the nth root of the mth power of a.* The fractional powers of complex numbers are discussed in Sec. 112.

**Examples.** $9^{\frac{3}{2}}=\sqrt{9^3}=27$, $\left(\frac{8}{27}\right)^{1\frac{1}{3}}=\left(\frac{8}{27}\right)^{\frac{4}{3}}=\sqrt[3]{\left(\frac{8}{27}\right)^4}$

$$=\frac{16}{81},\quad 3^{2\frac{1}{2}}=3^{\frac{5}{2}}=\sqrt{243}\approx 15.58.$$

**Note 1.** The base $a$ could be taken negative, but then its fractional powers might not result in real numbers. For instance,

$$(-2)^{\frac{3}{4}}=\sqrt[4]{(-2)^3}=\sqrt[4]{-8}$$

The root $\sqrt[4]{-8}$ cannot be a real number.

Ordinarily, elementary mathematics considers only positive bases of fractional powers.

**Note 2.** The exponents however, may be positive, negative or fractional. Negative exponents are of no less importance than positive exponents. In order to master logarithmic computations, it is necessary to do as many exercises as possible in order to get a proper understanding of negative and fractional exponents.

**Examples.** $9^{-\frac{3}{2}}=1:9^{\frac{3}{2}}=\frac{1}{27}$, $\left(\frac{8}{27}\right)^{-1\frac{2}{3}}=1:\left(\frac{8}{27}\right)^{1\frac{2}{3}}$

$$=\frac{243}{32},\ 3^{-2\frac{1}{2}}=1:3^{2\frac{1}{2}}=\frac{1}{\sqrt{243}}\approx 0.0642.$$

The introduction of fractional exponents does not involve any changes in the laws of exponents. Thus, the formula $a^m \cdot a^n = a^{m+n}$ and others are still valid.

**Example.** $a^{\frac{5}{7}} \cdot a^{-\frac{3}{7}} = a^{\frac{2}{7}}$. True enough, $a^{\frac{5}{7}} = \sqrt[7]{a^5}$; $a^{-\frac{3}{7}} = 1 : \sqrt[7]{a^3}$, $a^{\frac{2}{7}} = \sqrt[7]{a^2}$ so that our notation signifies $\sqrt[7]{a^5} \cdot \frac{1}{\sqrt[7]{a^3}} = \sqrt[7]{a^2}$ which is true (see Sec. 90, Rule 4).

### 126. The Method of Logarithms. Construction of Logarithmic Tables

The operations of multiplication, division, involution and evolution are much more labour-consuming than those of addition and subtraction, especially when they involve multidigit numbers. An insistent need for such operations arose in the 16th century in connection with the development of sea navigation which gave rise to improved astronomical observations and calculations. It was astronomical calculations that gave rise, at the turn of the 17th century, to computations by means of logarithms.

Today, such calculations are used whenever one has to do with large numbers. They are already useful when dealing with four-digit numbers and are absolutely necessary when five-place accuracy is required. Greater accuracy is rarely needed in practical situations.

The value of the logarithmic method consists in reducing multiplication and division of numbers to addition and subtraction which are much easier to perform. Likewise, involution and evolution and also a number of other computations (trigonometric, for instance) are greatly simplified.

The idea of the method can be illustrated in a few examples.

Suppose we have to multiply 10,000 by 100,000. The ordinary scheme of multiplying long numbers is not needed here at all: simply count the number of zeros in the multiplicand (4) and the multiplier (5) and add these numbers to get 9, which is the number of zeros in the product: 1,000,000,000 (9 zeros) This computation is legitimate because the factors are (integral) powers of 10: we multiply

$10^4$ by $10^5$ and the exponents are added. In the same way we perform division of powers of ten (division in this case is replaced by subtraction of the exponents).

But only a few numbers can be divided and multiplied in this manner. In the first million (excluding 1) we have only 6 such numbers, 10, 100, 1000, 10,000, 100,000, 1,000,000. Now there would be more numbers involved in multiplication and division if in place of the base 10 we took, say, 2, or some base closer to 1. For 2 as a base, we construct a table of the first 12 powers:

| Exponent (or logarithm): | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Power (or number): | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |

We will now use the term *logarithm* for the numbers (exponents) in the upper row, and the term *number* for the numbers (powers) in the lower row.

To multiply any two numbers of the lower row, add the two numbers above them in the upper row. To illustrate, we find the product of 32 by 64 by adding the 5 above 32 to the 6 above 64 to get 11. The answer is under 11: 2048. To divide 4096 by 256, take the numbers 12 and 8 above them; subtract 8 from 12 to get 4. The answer is under the number 4: 16. If we continue the table to the left and introduce zero and negative powers of the number 2, it is possible to perform division of small numbers by larger numbers.

Although there are fewer gaps between the powers of 2 than between the powers of 10, this table still involves only a few numbers and so is of little practical value. But if we take for the base a number much closer to 1 than the number 2, this defect will be overcome.

Let us take for the base the number 1.00001. There will be over a million (1,151, 292) successive powers of this number between 1 and 100,000. If we round off the values of these powers and retain only 6 significant digits, then a million rounded results will include all the whole numbers from 1 to 100,000. True, these will only be approximate values of the powers. But since in the multiplication and division of five-digit integers we are interested only in the first five places of the result, such tables will enable us to multiply, divide, etc. five-digit integers and, consequently, decimals with five significant digits.

That is precisely how the first tables of logarithms were constructed. * Their computation required many years of arduous labour. Today, anyone could do the job in about a month using the methods of higher mathematics. Three hundred years ago it required a lifetime. But the result was that many thousands of computors calculated many times faster through the use of these tables which were constructed once and for all time.

At the present time, logarithmic tables use the base 10 because this yields a number of computational advantages since our system of numeration is decimal. To obtain whole numbers it is necessary to use fractional powers of the number 10.

The logarithm of some number to the base 10 is called the *common logarithm*. Compilation of tables of common logarithms does not involve any particular difficulties if a table to the base 1.00001 has already been compiled. Indeed, suppose we want to find the common logarithm of the number 3, that is the exponent of the power to which we have to raise 10 in order to obtain 3. In the table to the base 1.00001 we find

$$10 \approx 1.00001^{230,258}$$

$$3 \approx 1.00001^{109,861}$$

Raising both members of the first equality to the power $\frac{1}{230,258}$, we get $1.00001 \approx 10^{(1:230,258)}$ and so the second equality can be written as $3 \approx 10^{(109,861:230,258)}$, which is to say that the logarithm of the number 3 to the base 10 is 0.47712. In the same way we can find the common logarithms of other numbers. **

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* In about 1590 by Bürgi of Switzerland, independently of him, and somewhat later the Scotsman Napier constructed a table based on a number very close to unity, but less than unity. Bürgi published his work in the year 1620, whereas Napier's tables appeared earlier, in 1614.

** The idea of constructing of table of logarithms to the base 10 belongs to the Scotsman Napier and the Englishman Briggs. Together they undertook to recalculate the earlier tables of Napier to the new base of 10. After the death of Napier, Briggs continued the work and published it completely in 1624. That is why the base—10 (common) logarithms are also called Briggsian. Fractional powers were not yet used in mathematics, but Napier and Briggs did without them since they defined the term logarithm somewhat differently from the presently accepted definition.

## 127. Basic Properties of Logarithms

*The logarithm of a number $N$ to a base $a$ is the exponent $x$ indicating the power to which $a$ must be raised to obtain $N$.*

**Notation:** $\log_a N = x$. Symbolically, $\log_a N = x$ is equivalent to $a^x = N$.

**Examples.** $\log_2 8 = 3$ since $2^3 = 8$; $\log_{1/2} 16 = -4$ since $\left(\frac{1}{2}\right)^{-4} = 16$; $\log_{1/2}\left(\frac{1}{8}\right) = 3$ since $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

From the definition of a logarithm follows the identity

$$a^{\log_a N} = N$$

**Examples.** $2^{\log_2 8} = 8$, i. e., $2^3 = 8$; $5^{\log_5 25} = 25$, $10^{\log_{10} N} = N$.

The numbers $a$ (the *logarithmic base*) and $N$ (the *number*) may be taken integral and fractional (see examples), but they must be positive if we want the logarithms to be real numbers.

The logarithms themselves may be negative. *Negative logarithms are just as important, practically, as positive logarithms.*

If for the base we take a number greater than unity (say, 10), then the larger number has a larger logarithm. The logarithms of numbers greater than unity are positive, those of numbers less than unity are negative. The logarithm of unity to any base is zero. The logarithm of a number equal to the base is always unity (in the common logarithms, $\log 10 = 1$). *

*The logarithm of a product is equal to the sum of the logarithms of the factors:*

$$\log_a (pq) = \log_a p + \log_a q$$

*The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor:*

$$\log_a \frac{p}{q} = \log_a p - \log_a q$$

*The logarithm of the power of a number is equal to the exponent times the logarithm of the number:*

$$\log_a p^m = m \log_a p$$

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* The number $a$ must not equal unity, otherwise numbers not equal to unity will not have a logarithm and any number will be the logarithm of unity.

We use the symbol $\log x$ to mean $\log_{10} x$ [translator].

*The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root:*

$$\log_a \sqrt[m]{p} = \frac{1}{m} \log_a p$$

(this is a consequence of the preceding property since $\sqrt[m]{p} = p^{\frac{1}{m}}$).

**Warning.** The logarithm of a sum is not equal to the sum of the logarithms; it is incorrect to write $\log_a p + \log_a q$ in place of $\log_a (p+q)$. This is a common mistake.

To take the logarithms of an expression means to express its logarithm in terms of the logarithms of the quantities that make up the expression.

**Examples of taking logarithms:**

$$(1)\ \log_a \frac{2p^2q}{\sqrt[3]{m^2}} = \log_a (2p^2qm^{-2/3})$$

$$= \log_a 2 + 2\log_a p + \log_a q - \frac{2}{3}\log_a m$$

$$(2)\ x = \frac{14.352 \cdot \sqrt{0.20600}}{185.06 \cdot 43110^2},$$

$$\log x = \log 14.352 + \frac{1}{2}\log 0.20600 - \log 185.06 - 2\log 43{,}110$$

Using a table of common logarithms, find $\log 14.352$, $\log 0.20600$, etc. and compute the right member of our equation; this is $\log x$. Then, using the table, find the number $x$ from its logarithm. For more details, see Secs. 131-134.

### 128. Natural Logarithms. The Number $e$

For practical purposes the most convenient are logarithms to the base 10 (common logarithms). In theoretical investigations however, it is more convenient to use a different base, namely the irrational number $e = 2.71828183$ (to eight decimal places). This amazing, at first glance, fact can only be explained in higher mathematics. Here we will merely show where the number came from. It is closely connected with the mode of computing logarithms that was explained in Sec. 126. When for the base we take a number $1 + \frac{1}{n}$

close to unity, say 1.00001 ($n = 100{,}000$), then for small numbers we get enormous logarithms such as 109,861 for the number 3. To make this logarithm of the same order of magnitude as 3, it has to be reduced by a factor of $n = 100{,}000$. Then it will be 1.09861. The number 3 will have the logarithm 1.09861 if for the base we take

$$\left(1+\frac{1}{n}\right)^n = 1.00001^{100,000} \text{ and not } 1+\frac{1}{n} = 1.00001$$

Indeed, we have

$$3 = (1.00001)^{109,861} = 1.00001^{100,000 \cdot 1.09861} = (1.00001^{100,000})^{1.09861}$$

Computing the quantity $1.00001^{100,000}$ to eight places of decimals, we get

$$\left(1+\frac{1}{n}\right)^n = 2.71826763 \quad (n = 100{,}000)$$

This number is very close to the number $e$; the first five digits coincide. If we took a number still closer to unity, say 1.000001 ($n = 1{,}000{,}000$), then reasoning as before, we would see that

$$\left(1+\frac{1}{n}\right)^n = 1.000001^{1,000,000}$$

is a still more convenient base.

To eight places of decimals, this number is 2.71828047. The first six digits are the same as those in the number $e$ and the seventh digit differs only by unity. The greater the number $n$, the less the number $\left(1+\frac{1}{n}\right)^n$ differs from the number $e$. In other words, *the number $e$ is the limit to which $\left(1+\frac{1}{n}\right)^n$ tends as $n$ increases without bound*. That is the definition of the number $e$.

We have seen that the base $1+\frac{1}{n}$ and, hence, $\left(1+\frac{1}{n}\right)^n$ as well, enables us to compute logarithms of all possible numbers the more exactly, the greater the number $n$. It is natural to expect that for this purpose the most convenient number is the limit to which $\left(1+\frac{1}{n}\right)^n$ tends as $n$ increases without bound, which is the number $e$. That precisely is the case. Computation of logarithms to the base $e$ can be per-

formed more quickly than to any other base. The methods for such calculation are given in higher mathematics.

The number $e$ itself can be expressed as a decimal to any desired degree of accuracy. Some tables contain approximate values of $e$ that far exceed any practical demands. However, it is impossible to express the number $e$ exactly by any decimal fraction or any rational fraction. What is more, $e$ is not only irrational, it is transcendental (see Sec. 91).

Logarithms taken to the base $e$ are called *natural logarithms*. Sometimes they are called Napierian, but this is wrong historically. *

**Notation.** Natural logarithms are usually denoted by $\ln x$ instead of $\log_e x$.

**Example.** $\ln 3 = 1.09861$.

To find the natural logarithm of a number $N$ from its common logarithm, divide the common logarithm of $N$ by the common logarithm of $e$ (which is equal to 0.43429...):

$$\ln N = \frac{\log N}{\log e} \approx \frac{\log N}{0.43429} \approx 2.30259 \log N$$

The quantity $\log e = 0.43429$ is called the *modulus of common* logarithms with respect to natural logarithms and is denoted by $M$, so that

$$\ln N = \frac{1}{M} \log N \text{ **}$$

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* The base actually used by Napier was the number $1 - 0.0000001$. If we wanted to reduce all logarithms of Napier's table by a factor of $10{,}000{,}000 = 10^7$ (see example analyzed above), we would have to take as the base the number $\left(1 - \frac{1}{k}\right)^k$, where $k = 10^7$, which could then be called the base of Napier's table. But this number is by no means equal to $e$ $\left(\text{it differs very slightly from the number } \frac{1}{e}\right)$.

** The rules given here for converting from natural logarithms to common logarithms and vice versa are particular cases of the general formulas

$$\log_a N = \log_b N \cdot \log_a b, \quad \log_a N = \frac{\log_b N}{\log_b a}$$

which permit converting from the logarithm of a number $N$ to the base $b$ to a logarithm of the same number to the base $a$. The second formula yields for $N = b$

$$\log_a b = \frac{1}{\log_b a}$$

**Example.** From the table of common logarithms we have $\log 2 = 0.30103$, whence

$$\ln 2 = \frac{1}{M} \cdot 0.30103 = 0.69315$$

To find the common logarithm of a number $N$ from a given natural logarithm of $N$, multiply the natural logarithm by the modulus of common logarithms with respect to natural logarithms: $M = \log e$:

$$\log N = \log e \ln N = M \ln N \approx 0.43429 \ln N$$

**Example.** $\ln 3 = 1.09861$, whence $\log 3 = M \cdot 1.09861 = 0.47712$.

Tables are provided to simplify multiplication by $M$ and $\frac{1}{M}$. They contain products of $M$ and $\frac{1}{M}$ by all one-digit or all two-digit factors. We give below a table for multiplication of $M$ and $\frac{1}{M}$ by single-digit numbers.

| | Multiples of $M$ | Multiples of $\frac{1}{M}$ |
|---|---|---|
| 1 | 0.43429 | 2.30259 |
| 2 | 0.86859 | 4.60517 |
| 3 | 1.30288 | 6.90776 |
| 4 | 1.73718 | 9.21034 |
| 5 | 2.17147 | 11.51293 |
| 6 | 2.60577 | 13.81551 |
| 7 | 3.04006 | 16.11810 |
| 8 | 3.47436 | 18.42068 |
| 9 | 3.90865 | 20.72327 |

### 129. Common Logarithms

From now on, we shall simply use the word logarithm in the meaning of common logarithm.

The logarithm of unity is zero.

The logarithms of 10, 100, 1000, etc., are 1, 2, 3, and so on, that is, they have as many positive units as there are zeros following unity.

The logarithms of the numbers 0.1, 0.01, 0.001, etc. are equal to $-1$, $-2$, $-3$, etc.; hence they have as many negative units as there are zeros (including the zero in the units place) preceding unity.

The logarithms of the other numbers have a fractional part which is called the *mantissa*. The integral (whole-number) portion of the logarithm is termed the *characteristic*.

Numbers greater than unity have positive logarithms. Positive numbers less than unity have negative logarithms (negative numbers do not have real logarithms).

For example, * $\log 0.5 = -0.30103$, $\log 0.005 = -2.30103$.

For the sake of convenience in locating the logarithm from a number or the number from the logarithm, negative logarithms are not given in this natural form but in an "artificial" form. In the so-called artificial form, a negative logarithm *has a positive mantissa and a negative characteristic.*

For example, $\log 0.005 = 3.69897$. This notation means that $\log 0.005 = -3 + 0.69897 = -2.30103$.

To transfer a negative logarithm from its natural form to the artificial form, (1) increase the absolute value of its characteristic by unity, (2) put the minus sign (a bar) over that number, (3) subtract from 9 all digits of the mantissa except the last nonzero digit; subtract the last nonzero digit from 10. The differences obtained are written in the same places of the mantissa as the digits being subtracted. Zeros at the end remain unchanged.

**Example 1.** Reduce $\log 0.05 = -1.30103$ to the artificial form: (1) increase by 1 the absolute value of the characteristic (which is 1); this yields 2, (2) write the characteristic in artificial form as $\bar{2}$ and place the decimal point, (3) subtract the first digit of the mantissa (which is 3) from 9; this yields 6; write the 6 in the first decimal place. The subsequent places will have the digits 9 (= 9 − 0), 8 (= 9 − 1), 9 (= 9 − 0) and 7 (= 10 − 3), or

$$-1.30103 = \bar{2}.69897$$

**Example 2.** Represent −0.18350 in artificial form: (1) increase 0 by 1 to get 1, (2) we have $\bar{1}$, (3) subtract from 9 the digits 1, 8, 3, and from 10 the digit 5; the zero at the end remains unchanged. This yields

$$-0.18350 = \bar{1}.81650$$

To transfer a negative logarithm from the artificial to the natural form, (1) reduce the absolute value of its characteristic by unity, (2) place the minus sign to the left of the number, (3) handle the digits of the mantissa as in the preceding case.

* All subsequent equations are approximate to within one half unit of the last digit.

**Example 3.** Represent $\bar{4}.68900$ in natural form. (1) $4-1=3$, (2) we have $-3$, (3) subtract the digits 6, 8 of the mantissa from 9, and the digit 9 from 10; the two zeros at the end remain unchanged. This yields

$$\bar{4}.68900 = -3.31100$$

### 130. Operations Involving Artificial Expressions of Negative Logarithms

In the case of artificial expressions of logarithms there is no need to convert them to the natural form from the start. With a little practice, the techniques described below enable one to handle artificial expressions directly just as fast as natural ones.

**Addition.** Mantissas are added as usual; after adding the tenths it may happen that one or more units have to be carried; if that is the case, then the carried digit is added to the positive characteristics when adding the characteristics (which may include both positive and negative numbers).

**Example 1.** $\bar{1}.17350+2.88694+\bar{3}.99206$.

**Work:**

$$\begin{array}{r} {\scriptstyle 2\ 2111} \\ \bar{1}.17350 \\ +2.88694 \\ \bar{3}.99206 \\ \hline 0.05250 \end{array}$$

Here, adding the tenths yields $2+1+8+9=20$ (the carried digits are written at the top of the proper column). Write down 0 and carry the 2. Adding the characteristics yields $2+\bar{1}+2+\bar{3}=0$.

**Example 2.**

$$\begin{array}{r} {\scriptstyle 1\ \ 11} \\ 2.7458 \\ +\bar{4}.3089 \\ \hline \bar{1}.0547 \end{array}$$

Adding the characteristics here we get $1+2+\bar{4}=\bar{1}$.

**Subtraction.** The mantissa of the subtrahend is subtracted column by column from the mantissa of the minuend both when the former is less than the latter and vice versa. In the latter case, for the tenths digit of the minuend we borrow a positive unit from the characteristic.

**Example 1.**

$$\begin{array}{r} {\scriptstyle \cdot\ \cdot\ \cdot\ \cdot} \\ \bar{2}.1741 \\ 5.1846 \\ \hline \bar{2}.9895 \end{array}$$

In subtracting tenths we had to borrow a positive unit from the characteristic $\bar{2}$, which changed it to 3. Subtraction of the characteristics yields $\bar{3}-5=2$.

**Example 2.**

$$\begin{array}{r} \bar{1}.2080 \\ -3.1916 \\ \hline \bar{4}.0164 \end{array}$$

Here we do not have to borrow from the characteristic; $\bar{1}-3=\bar{4}$.

**Example 3.**

$$\begin{array}{r} 0.1265 \\ -1.9371 \\ \hline \bar{2}.1894 \end{array}$$

Here we can see that even when subtracting a positive logarithm from a positive logarithm, the result can be obtained directly in artificial form. It is advisable to do so.

In *joint addition and subtraction*, it is sometimes preferable to replace all subtractions by additions. In this case, if the subtrahend is a positive number, the corresponding negative addend is converted to artificial form. But if it is negative number specified in artificial form, it is converted to natural form and the minus sign is dropped. The resulting addends may then be called *cologarithmic addends*.

**Example.** $0.1535-\bar{1}.1236+\bar{1}.1686-4.3009=0.1535+$colog. addend $\bar{1}.1236+\bar{1}.1686+$colog. addend $4.3009=0.1535+0.8764+\bar{1}.1686+\bar{5}.6991=\bar{5}.8976$.

**Work:**

$$\begin{array}{r} 0.1535=0.1535 \\ -\bar{1}.1236=0.8764 \\ +\bar{1}.1686=\bar{1}.1686 \\ -4.3009=\bar{5}.6991 \\ \hline \bar{5}.8976 \end{array}$$

**Multiplication.** To multiply an artificial logarithm by a positive number, multiply separately first the mantissa and then the characteristic; if the multiplier is a one-digit number, then the number of positive units obtained in multiplying the mantissa is immediately added to the negative product of the multiplier into the characteristic. However, if the multiplier is multi-digit, carry the multiplication by the mantissa to the end and add the product of the multiplier by the characteristic.

**Example 1.**

$$\begin{array}{r} {\scriptstyle 3\ 264} \\ \bar{5}.4397 \\ \times \quad 7 \\ \hline \bar{39}.0779 \end{array}$$

**Example 2.**

$$\begin{array}{r} \bar{1}.4397 \\ \times \quad 17 \\ \hline 4397 \\ 3078 \\ \hline 7.475 \\ -17 \\ \hline \bar{1}0.475 \end{array}$$

Use the rules of short-cut multiplication; see Sec. 55.

If it is necessary to multiply a negative logarithm in artificial form by a negative number, it is best to convert the logarithm to natural form first.

**Division.** If the divisor is a negative or multi-digit positive number, it is best to convert to natural form. If the divisor is a single-digit positive number, leave the dividend in artificial form. If the characteristic is exactly divisible, then divide separately the characteristic and then the mantissa. If the characteristic is not exactly divisible, then add to it mentally a least number of negative units such that the resulting number is exactly divisible; then add mentally to the mantissa the same number of positive units.

**Example.** $\bar{2}.5638:6 = \bar{1}.7606$. So as to make the characteristic divisible by 6, add 4 negative units. Dividing the result, $-6$, by 6 yields $-1$. Now first add 4 positive units to the mantissa and then divide 4.5638 by 6.

### 131. Finding the Logarithm of a Number

The logarithms of integral powers of 10 are found without tables (Sec. 129). To find the logarithm of any other number, do as follows.

(A) *Finding the characteristic.* For number greater than unity, the characteristic is equal to the number of digits in the integral portion minus one.

**Examples.** $\log 35.28 = 1$ (characteristic); $\log 3.528 = 0$ (characteristic); $\log 60{,}100 = 4$ (characteristic). The numbers thus found in these examples are then followed by a decimal point and the digits of the mantissa.

For numbers less than unity, the characteristic of the artificial form of the logarithm is equal to the number of zeros preceding the significant digits of the number (including the zero in the units place).

**Examples.** $\log 0.00635 = \bar{3}$ (characteristic); $\log 0.1002 = \bar{1}$ (characteristic); $\log 0.06004 =$ (characteristic).

(B) *Finding the mantissa.* To find the mantissa of a decimal fraction (whether pure or mixed), drop the decimal point and enter the table to find the mantissa of the resultant whole number. To do this, drop all zeros (if there are any) at the end of the whole number. For example, the mantissa of the number 20.73 is equal to the mantissa of the number 2073, the mantissa of the number 6,004,800 is equal to the mantissa of the number 60,048.

When using four-place tables of logarithms, we leave only the first four digits of any whole number; if the tables are five-place we use the first five digits. The other digits are discarded because they do not affect (for all practical purposes, at any rate) the digits of the mantissa given in the table.

A four-place table gives the mantissa of a three-digit number directly; a five-place table gives the mantissa of a four-digit number directly. The mantissas of four- (five-) digit numbers are found by adding so-called mean differences, or proportional parts (see examples given below).

*A four-place table of logarithms* is given on pages 18-22.

**Example 1.** Find the logarithm of the number 45.8. We find the characteristic by inspection (without the use of the table): 1. Dropping the decimal point, we get the whole number $N = 458$. Taking the first two digits (45), move along row 45 to column 8 to find 6609. This is the mantissa. We thus have $\log 45.8 = 1.6609$.

**Example 2.** Find $\log 0.02647$. We obtain the characteristic by inspection, without the table: $\bar{2}$. Dropping the decimal point we have the number 2647. Taking its first two digits (26) we move along row 26 up to column 4 (4 is the third digit of the given number) and read 4216. This is the mantissa of $\log 264$. In the "proportional parts" of the table we find the correction corresponding to the digit 7 (the fourth digit of the given number). It is in row 26, column 7 of proportional parts and yields 11. Add this to the earlier obtained mantissa to get $4216 + 11 = 4227$. This is the mantissa of the given number. We thus have $\log 0.02647 = \bar{2}.4227$.

**Work:**

$$\begin{array}{rl} \log 0.0264 = & \bar{2}.4216 \\ 7 & +11 \\ \hline \log 0.02647 = & \bar{2}.4227 \end{array}$$

**Note.** The corrections obtained in the proportional parts data of the table are computed by means of interpolation (see Sec. 64). The use of interpolation simplifies the work of

the computor. From the table we can see that the mantissa of 2640 is less than the mantissa of 2650 by 4232 — 4216 = 16 (decimal parts). The difference of 10 between the two numbers corresponds to a difference of 16 between the mantissas. Working the proportion, we get

$$x : 16 = 7 : 10 = 0.7, \quad x = 16 \cdot 0.7 = 11$$

*Five-place table of logarithms.*

**Example 1.** Find $\log 0.02647$. We find the characteristic by inspection: $\log 0.02647 = \bar{2}$. Dropping the decimal point, we have the number 2647. Find row 264 and go along it to column 7 to find 275. These represent the last three digits of the mantissa. The first two (42) are found at the beginning of the row. The entire mantissa is 42275; $\log 0.02647 = \bar{2}.42275$.

In most rows, the first two digits are not indicated. They are taken from the succeeding row (if there is an asterisk in front of the last three digits of the mantissa) or from the preceding row (if there is no asterisk).

**Example 2.** Find $\log 6764$. The characteristic is 3. Take row 676 of the table of logarithms and go along it to column 4 to get the last three digits of the mantissa: 020. They have an asterisk and so the first two digits (83) are taken from the next lower row 677. The entire mantissa is 83020; $\log 6764 = 3.83020$.

**Example 3.** Find $\log 6.6094$. Find the characteristic by inspection. It is 0. Drop the decimal point to get 66094. In row 660 (the first three digits of the number) we seek column 9 (the fourth digit) and find the number 014 with an asterisk. These are the last three digits of the mantissa of the number 6609. The first two (82) are found in the next row. The mantissa of $\log 6609$ is 82014. Find the correction corresponding to the last digit, 4, of the given number. In the column *PP* we find a table headed by 6 ($d = 6$ is the difference between the mantissas of the numbers 6609 and 6610). We find the number 4 in the left part of this table. Opposite 4 is 2.4, which we round off to 2. This is the correction which we add to the earlier found mantissa to get $82014 + 2 = 82016$; $\log 6.6094 = 0.82016$.

**Work:**

$$\begin{array}{r} \log 6.609 = 0.82014 \\ 4 \quad + \quad 2 \\ \hline \log 6.6094 = 0.82016 \end{array}$$

## 132. Finding a Number from a Logarithm *.

Disregard the characteristic and seek, in the table, the given mantissa or one close to it. This is used to find a certain whole number (it is found directly in the first case and with the aid of a correction, in the second; see the examples). Then examine the characteristic. If it is zero or positive, then the integral part is formed by taking one unit more than the number of units of the characteristic (zeros can be annexed at the end of the number if necessary). If the characteristic is negative, then place before the number found as many zeros as there are negative units in the characteristic; the zero on the left is set off by a decimal point. The number thus found corresponds to the given logarithm.

*Four-place table* (see pages 18-22).

**Example 1.** Find the number whose logarithm is equal to 3.4683 (that is, the number $10^{3.4683}$). First in the table seek the mantissa 4683 or one close to it. Run down one of the columns, say column 0, and seek a number whose first two digits are 46 or a number close to 46. We find such a number (4624) in row 29. Thereabouts look for the mantissa 4683; we find it in row 29, column 4. Hence, the number with mantissa 4683 is 294. Since the characteristic 3 is positive, we take $3+1=4$ digits for the integral part. And so we annex a zero at the end of 294. This yields $3.4683 = \log 2940$.

**Example 2.** Find the number whose logarithm is 3.3916. Proceeding as in the previous example, we do not find the number 3916 among the mantissas, but we find a close number, 3909, at the intersection of row 24 and column 6. Thus, the number 246 corresponds to the mantissa 3909, which yields the first three significant digits of the desired number. The fourth digit is found by computing the correction. The given mantissa 3916 exceeds the tabular value 3909 by 7. We seek this digit in the same row 24 in "Proportional Parts". It is found in column 4. The digit 4 is thus the fourth significant digit of the desired number; the number 2464 corresponds to the mantissa 3916. Now examine the characteristic. Since it is negative and contains three units, we put three zeros in front of the number we found and set off one decimal place. We thus have $3.3916 = \log 0.002464$.

---

* When looking up a number on the basis of its four-place logarithm, it is best to use a table of antilogarithms (see Sec. 133). It is not advisable for five-place work to double the volume of a logarithmic table by adjoining a table of antilogarithms.

**Work:**

$$\log x = \bar{3}.3916$$

| | | | |
|---|---|---|---|
| 3909 | log 246 | | |
| +7 | | 4; | $x = 0.002464$ |
| 3916 | log 2464 | | |

**Note 1.** Bear in mind firmly that when seeking a number on the basis of a logarithm the proportional parts correction is annexed to it and not added to the last digit.

**Note 2.** Do not forget that the correction is to be sought in the same row as the number that is our approximation to the mantissa. If this row does not have the correction of the mantissa we need, then take the next closest correction.

*Five-place table of logarithms.*

**Example 1.** Find the number whose logarithm is $\bar{2}.43377$. Turn the pages noting the first two digits of the mantissas (the numbers increase). Find 43 and then in the vicinity look for the last three digits, or 377. These digits are located at the intersection of row 271 and column 5. The number with the mantissa 43377 is thus 2715. Taking into account the characteristic ($\bar{2}$), we have $\bar{2}.43377 = \log 0.02715$.

**Note.** In most cases, the last three digits of a mantissa are found in the same row as the first two digits, or in one of the rows below; it is, however, possible that the last three digits will be the nearest row above, in which case they are preceded by an asterisk.

**Example 2.** Find a number whose logarithm is 0.14185. Proceed in the same manner as in the above example; we do not find 14185 among the mantissas but we do find 14176, which is close. The last three digits of the mantissa (176) lie above the first two and so we find an asterisk in front. The mantissa 14176, which stands at the intersection of row 138 and column 6, is associated with the number 1386, which yields the first four digits of the desired number. The fifth digit is computed by interpolation. The mantissa at hand exceeds the tabular value by $185 - 176 = 9$. Now the difference between the two closest tabular mantissas is $208 - 176 = 32$.

In the column *PP* we find a small table headed 32. In it, to the right, we look for a number close to 9 and find 9.6. Opposite this number we find 3. This digit is the fifth significant digit of the desired number. The number having the mantissa 14185 is 13863. Taking into account the characteristic, we get $0.14185 = \log 1.3863$.

**Work:**

$$\begin{array}{rll} \log x = 0.14185 & & \\ \quad 14176 & 1386 & \\ \quad +9 & 3; & x = 1.3863 \\ \hline \quad 14185 & 13863 & \end{array}$$

**Note.** When finding a number from a logarithm, the proportional parts data are annexed to the number and are not added to the last digit.

### 133. Tables of antilogarithms

The so-called table of antilogarithms (see pages 23-27) is the same as a logarithmic table only the data are arranged differently so as to simplify finding a number from a given logarithm. Only mantissas (denoted by $m$) are given in the table (in boldface type). If the mantissa has three decimal places, the table gives a whole number directly; if the mantissa has four decimal places, the number is found with the aid of proportional parts data (see examples). The given characteristic is then inspected. If it is zero or positive, then the integral part has one more digit than the number of units in the characteristic (any number of required zeros may be annexed at the end of the number). If the characteristic is negative, then the number is preceded by as many zeros as there are units in the characteristic. The zero at the extreme left is set off by a decimal point. The number thus found corresponds to the given logarithm.

**Example 1.** Find a number whose logarithm is equal to 2.732 (that is, the number $10^{2.732}$). Disregard the characteristic and take the first two digits of the mantissa (73). Go along row 73 up to column 2 to find the number 5395. Since the characteristic 2 is positive, the integral part has $2+1=3$ digits. Our answer is $10^{2.732}=539.5$.

**Example 2.** Given $\log x = \bar{3}.2758$. Find $x$. Disregard the characteristic and find the number in row 27 and column 5. It is 1884. Find the digit in the proportional parts data corresponding to the digit 8. It is 3. Add it to the number that was found: $1884+3=1887$. Inspect the characteristic. Since it is negative and contains three units, we put three zeros in front of the number 1887 and set off one decimal place from the left. We have

$$x = 0.001887 \text{ or } \log 0.001887 = \bar{3}.2758$$

**Work:**

$$\log x = 3.2758$$

| | |
|---|---|
| 275 | 1884 |
| 8+ | 3 |
| 2758 | 1887 |

$$x = 0.001887$$

**Example 3.** $\log x = 0.0817$. Find $x$.

| | |
|---|---|
| 081 | 1205 |
| 7+ | 2 |
| 0817 | 1207 |

$$x = 1.207$$

**Note.** When finding a number from a logarithm with the aid of antilog tables, the proportional parts data are always added to the last digit and not annexed.

### 134. Logarithmic Computations (Worked Examples)

**Example 1.** Compute $u = \frac{ab}{\sqrt{a^2 - b^2}}$ where $a = 4.352$, $b = 1.800$.

(1) Taking logs,

$$\log u = \log \frac{ab}{\sqrt{a^2 - b^2}} = \log \frac{ab}{\sqrt{(a+b)(a-b)}}$$

$$= \log a + \log b - \tfrac{1}{2}[\log (a+b) + \log (a-b)]$$

(2) We find $a+b$ and $a-b$:

| | |
|---|---|
| $+\ a = 4.352$ | $-\ a = 4.352$ |
| $b = 1.800$ | $b = 1.800$ |
| $a+b = 6.152$ | $a-b = 2.552$ |

(3) First compute $\log a + \log b$, then $\frac{1}{2}[\log (a+b) + \log (a-b)]$:

$$\log a = \log 4.352 = 0.6387$$
$$\log b = \log 1.800 = 0.2553$$
$$\log a + \log b = 0.8940$$
$$\log (a+b) = \log 6.152 = 0.7890$$
$$\log (a-b) = \log 2.552 = 0.4068$$
$$\log (a+b) + \log (a-b) = 1.1958$$
$$\tfrac{1}{2}[\log (a+b) + \log (a-b)] = 0.5979$$

(4) Find $\log u$ and then $u$:

$$\begin{array}{r} 0.8940 \\ -\ 0.5979 \\ \hline \log u = 0.2961, \end{array} \quad u = 1.977$$

**Example 2.** Evaluate $P = pe^{-\frac{k}{p}h}$ where $p = 10.33$, $k = 0.00129$, $h = 1000$, and $e$ is the base of natural logarithms ($e \approx 2.7183$).

(1) $\log P = \log p - \frac{k}{p} h \log e = \log p - \frac{k}{p} hM$,

where $M = \log e \approx 0.4343$ (the modulus of common logarithms with respect to natural logarithms; see Sec. 128).

(2) Find $\log p$:

$$\log p = \log 10.33 = 1.0141$$

(3) Take the logarithms of the expression $\frac{k}{p} hM$:

$$\log \frac{k}{p} hM = \log k + \log h + \log M - \log p$$

(4) Evaluate this logarithmic expression:

$$\begin{array}{rl} \log k = \log 0.00129 & = \bar{3}.1106 \\ \log h = \log 1000 & = 3.0000 \\ \log M = \log 0.4343 & = \bar{1}.6378 \\ \operatorname{colog} p = \operatorname{colog} 10.33 & = \bar{2}.9859 \\ \hline \log \frac{k}{p} hM = & \bar{2}.7343 \end{array}$$

Whence $\frac{k}{p} hM = 0.05424$.

(5) Evaluate $\log P$ (see Item 1) and then $P$:

$$\begin{array}{rl} \log p & = 1.0141 \\ -\ \frac{k}{p} hM & = 0.0542 \\ \hline \log P = 0.9599, \text{ whence } & P = 9.118 \end{array}$$

### 135. Combinatorics (Permutations and Combinations)

There are two basic ways of arranging items chosen from some set of distinct objects (elements). They are termed *permutations* and *combinations*.

**1. Permutations.** Take $m$ distinct objects (elements) $a_1$, $a_2$, ..., $a_m$. Rearrange these elements in all possible ways keeping their number constant and only changing their order. Each such arrangement (including the original one) is called a *permutation*. The total number of permutations of $m$ elements is denoted by $P_m$. This number is equal to the product of all integers from 1 (or, what is the same thing, from 2) to $m$ inclusive:

$$P_m = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (m-1) \cdot m = m! \qquad (1)$$

The symbol $m!$ (read "*m factorial*") denotes the product $1 \cdot 2 \cdot 3 \cdot \ldots \cdot (m-1) \cdot m$.

**Example 1.** Find the number of permutations of three elements $a$, $b$, $c$. We have $P_3 = 1 \cdot 2 \cdot 3 = 6$. Indeed, we have 6 permutations:

(1) *abc*, (2) *acb*, (3) *bac*, (4) *bca*, (5) *cab*, (6) *cba*.

**Example 2.** In how many ways can five posts be assigned to five persons elected to the administrative board of a sports club? Write a list of the posts and opposite each write a name. This will be one permutation. The total number of such permutations is $P_5 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

**Note.** When $m = 1$ in the expression $1 \cdot 2 \cdot 3 \cdot \ldots \cdot m$, there is only one number, 1. It is therefore agreed (by definition) that $1! = 1$. For $m = 0$ the expression $1 \cdot 2 \cdot 3 \cdot \ldots \cdot m$ becomes meaningless, and so it is defined that $0! = 1$. See Item 2 below for the justification of this convention.

Let us now take this set of $m$ elements and make up groups of $n$ elements in each, arranging the $n$ elements in different order. The resulting arrangements are termed *permutations* of $m$ elements taken $n$ at a time. The total number of permutations of $m$ elements $n$ at a time is denoted by $P_m^n$ (or ${}_nP_m$). This number is equal to the product of $n$ successive integers of which the largest is $m$:

$$P_m^n = m\,(m-1)\,(m-2)\,\ldots\,[m-(n-1)] \qquad (2)$$

**Example 3.** Find the number of permutations of four elements *abcd* taken two at a time. We have $P_4^2 = 4 \cdot 3 = 12$. These permutations are the following:

*ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc*

**Example 4.** A meeting elects 8 persons to its supreme body, which then chooses a chairman, a secretary and a treasurer. In how many ways can these posts be assigned?

The desired number is the number of permutations of 8 elements taken 3 at a time, or $P_8^3 = 8 \cdot 7 \cdot 6 = 336$.

**2. Combinations.** Take $m$ distinct elements and make up groups of $n$ elements in each but disregard the order of the elements in each group. We then have *combinations* of $m$ elements taken $n$ at a time.

The total number of distinct combinations is denoted by $C_m^n$. This number (which is integral of course) can be represented by the formula * (see Item 1)

$$C_m^n = \frac{P_m}{P_n P_{m-n}} = \frac{m!}{n!\,(m-n)!} \tag{3}$$

By definition we assume $C_m^0 = 1$ [this value is obtained from (3)].

The expression $\frac{m!}{n!\,(m-n)!}$ is often abbreviated to $\binom{m}{n}$.

It is clear that $\binom{m}{n} = \binom{m}{m-n}$, that is, $C_m^n = C_m^{m-n}$.

In computations it is often more convenient to make use of other expressions for the number of combinations, namely

$$C_m^n = \frac{P_m^n}{P_n} = \frac{m\,(m-1)\ldots[m-(n-1)]}{1 \cdot 2 \cdot \ldots \cdot n}$$

or

$$C_m^n = \frac{P_m^{m-n}}{P_{m-n}} = \frac{m\,(m-1)\ldots(n+1)}{1 \cdot 2 \ldots (m-n)}$$

**Example 5.** Find all the combinations of five elements *abcde* taken three at a time. We have $C_5^3 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10$. These ten combinations are

*abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde*

**Example 6.** There are eight candidates for three counters. In how many ways can the assignments be made? Since the duties of each counter are the same, we have combinations and not permutations, as in Example 4 of Item 1. The sought-for number is

$$C_8^3 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

Combinatorial mathematics deals with many more types of arrangements than those described above. One of the most

* There is only one combination containing all $m$ elements and so $C_m^m = 1$. Formula (3) gives this value only if we agree that $0! = 1$.

important types is *permutations with repetitions of the elements*. These are defined as follows. Take $m$ elements, of which $m_1$ are identical elements of type one, $m_2$ are identical elements of a second kind, etc. Permute them in all possible ways. We then get *permutations with repetitions*. The number of distinct permutations with repetitions is

$$\frac{P_m}{P_{m_1}P_{m_2}\ldots P_{m_k}} \quad \text{or} \quad \frac{m!}{m_1!\,m_2!\ldots m_k!}$$

($m_1+m_2+\ldots+m_k=m$, and $k$ is the number of kinds).

**Example 7.** Find the number of distinct permutations of the letters *aaabbcc* with repetitions of the elements. Interchanging the first two letters does not result in a new arrangement. The same occurs when we interchange the fourth and fifth letters, and in other cases. But the arrangements *abaabcc*, *caaabcb* and certain others are distinctly new ones. In this example, $m_1=3$, $m_2=2$, $m_3=2$; $m=m_1+m_2+m_3=7$. The number of distinct permutations is

$$\frac{7!}{3!\,2!\,2!}=\frac{2\cdot3\cdot4\cdot5\cdot6\cdot7}{2\cdot3\cdot2\cdot2}=210$$

**Example 8.** Find the number of distinct permutations made up of the signs $+++\,---$. Here, $m_1=4$, $m_2=3$; $m=m_1+m_2=7$. The desired number is $\frac{7!}{4!\,3!}=35$. From this example it is easy to see that the number of permutations of $m$ elements, among which are repeated $m_1$ elements of the first kind and $m_2$ elements of the second kind, is equal to the number of combinations of $m$ elements taken $m_1$ at a time, or to the number of combinations of $m$ elements taken $m_2$ at a time. Indeed, each permutation is associated with one and only one selection of positions with the $+$ sign. Thus, in the permutation $++--+-+$ the $+$ signs occupy the 1, 2, 5, 7 positions so that the corresponding combination is 1, 2, 5, 7. Hence, there are as many permutations as there are distinct combinations of seven numbers taken four at a time.

### 136. The Binomial Theorem

*Newton's binomial theorem* states a formula expressing $(a+b)^n$ in the form of a polynomial for positive integral $n$*.

* The term "Newton's binomial theorem" is a misnomer, firstly because $(a+b)^n$ is not a binomial, secondly, the expansion of $(a+b)^n$ for positive integral $n$ was known before Newton's time. But to Newton goes the credit for the bold and extremely fruitful idea of extending the expansion to the case of negative and fractional $n$.

The binomial formula for positive integral $n$ is

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots + \binom{n}{n-1} a b^{n-1} + b^n \quad (1)$$

or, what is the same thing (see p. 242),

$$(a+b)^n = a^n + \frac{n!}{1!\,(n-1)!} a^{n-1} b + \frac{n!}{2!\,(n-2)!} a^{n-2} b^2 + \ldots \quad (2)$$

Accordingly, it is assumed that $\binom{n}{0} = \binom{n}{n} = 1$ and also $0! = 1$ (see note on page 242). With this convention, the first and last terms of the expansion are of the same form as the other terms.

For computational purposes it is more convenient to use the formula

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} a^{n-3} b^3 + \ldots + b^n \quad (3)$$

**Example 1.** $(a+b)^3 = a^3 + 3a^2 b + \frac{3\cdot 2}{1\cdot 2} ab^2 + b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$.

**Example 2.** $(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.

The numbers $1, n, \frac{n(n-1)}{1\cdot 2}, \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$, etc. are termed *binomial coefficients*. They may be obtained in the following manner using only addition. In the top row (see accompanying array) write two units. All other rows begin and end with unity. The intermediate numbers are found by adding adjacent numbers of the row above. Thus, the number 2 in the second row is found by adding the two units of the first row; the third row is obtained from the second this way: $1+2=3$, $2+1=3$; the fourth row from the third as follows: $1+3=4$, $3+3=6$, $3+1=4$, and so forth. The numbers in one row are the binomial coefficients of an appropriate power. This array is called *Pascal's triangle* (or the arithmetic triangle).

```
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

## The Binomial Theorem for Fractional and Negative Exponents

Suppose we have the expression $(a+b)^n$ where $n$ is a fractional or negative number. Let $|a| > |b|$. Represent $(a+b)^n$ as $a^n(1+x)^n$. The quantity $x=\frac{b}{a}$; its absolute value is less than unity. The expression $(1+x)^n$ may be computed to any desired degree of accuracy by formula (3).

**Example 1.** $\frac{1}{1+x}=(1+x)^{-1}$. Here $n=-1$.

Since $\frac{n(n-1)}{1\cdot 2}=\frac{(-1)\cdot(-2)}{1\cdot 2}=1$,

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}=\frac{(-1)\cdot(-2)\cdot(-3)}{1\cdot 2\cdot 3}=-1$$

and so on, we have $(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots$.

The number of terms in the right-hand member is infinitely great, but when $|x|<1$, the sum of the terms, as their number increases without bound, tends to the limit $\frac{1}{1+x}$ (the expression in the right member, if $|x|<1$, is an infinitely decreasing geometric series).

**Example 2.** Compute $\sqrt{1.06}$ to five decimal places.

Represent $\sqrt{1.06}$ as $(1+0.06)^{\frac{1}{2}}$ and apply formula (3):

$$(1+0.06)^{\frac{1}{2}}=1+\tfrac{1}{2}\cdot 0.06+\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{1\cdot 2}\cdot 0.06^2$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\cdot 2\cdot 3}\cdot 0.06^3+\ldots=1+0.03-0.00045$$

$$+0.0000135-\ldots$$

Subsequent terms do not affect the first five decimal places and so, summing the four terms written out, we have

$$\sqrt{1.06}\approx 1.02956$$

**Example 3.** Find five significant figures of the number $\sqrt[3]{130}$.

The cube nearest to 130 is $125 = 5^3$. Represent $\sqrt[3]{130}$ as $(125+5)^{\frac{1}{3}} = 125^{\frac{1}{3}}(1+0.04)^{\frac{1}{3}} = 5(1+0.04)^{\frac{1}{3}}$. Carry the answer to seven places (taking note of the fact that the error is accumulative in addition and is then increased fivefold):

$$(1+0.04)^{\frac{1}{3}} = 1+\frac{1}{3}\cdot 0.04+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1\cdot 2}\cdot 0.04^2$$

$$+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1\cdot 2\cdot 3}\cdot 0.04^3+\ldots$$

$$= 1+0.0133333-0.0001778+0.0000040-\ldots = 1.0131595$$

The discarded terms do not affect the seventh digit and we find $5\cdot 1.0131595 = 5.0657975$. Accurate to the fifth decimal place, we have $\sqrt[3]{130} = 5.06580$. A more precise computation (with regard for the next term) yields 5.0657970, all decimals correct.

Using this device we can extract the root of any degree of any number rapidly and to any desired degree of accuracy.

### Generalized Binomial Formula

$$(a_1+a_2+a_3+\ldots+a_k)^n = \sum \frac{n!}{n_1!n_2!\ldots n_k!}\, a_1^{n_1} a_2^{n_2}\ldots a_k^{n_k}$$

where $n$ is a positive integer.

The symbol $\sum$ signifies that we must take the sum of all terms of the form

$$\frac{n!}{n_1!n_2!\ldots n_k!}\, a_1^{n_1} a_2^{n_2}\ldots a_k^{n_k}$$

where $n$ is the given exponent and $n_1, n_2, \ldots, n_k$ are arbitrary integers or zeros whose sum is equal to $n$. The number 0! is taken equal to 1.

**Example**

$$(a+b+c+d)^3 = \sum \frac{3}{n_1!\, n_2!\, n_3!\, n_4!}\, a^{n_1} b^{n_2} c^{n_3} d^{n_4}$$

the number $n=3$ may be given as the sum of $k=4$ integral sumands thus:

$$3=3+0+0+0,$$
$$3=2+1+0+0,$$
$$3=1+1+1+0$$

Accordingly, we have

$$(a+b+c+d)^3=\frac{31}{31\,01\,01\,01}(a^3b^0c^0d^0+a^0b^3c^0d^0$$
$$+a^0b^0c^3d^0+a^0b^0c^0d^3)+\frac{31}{21\,11\,01\,01}(a^2bc^0d^0+ab^2c^0d^0$$
$$+a^2b^0cd^0+ab^0c^2d^0+\ldots)$$
$$+\frac{31}{11\,11\,11\,01}(abcd^0+abc^0d+ab^0cd+a^0bcd)$$
$$=a^3+b^3+c^3+d^3+3(a^2b+ab^2+a^2c+ac^2+a^2d+ad^2$$
$$+b^2c+bc^2+b^2d+bd^2+c^2d+cd^2)+6(abc+abd+acd+bcd)$$

## Properties of Binomial Coefficients

1. The coefficients of terms the same distance from the ends of the expansion are the same.

For example, in the expansion

$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$

the coefficients of the second and second to the last terms are the same, 6; the coefficients of the third term from the beginning and the third from the end are the same, 15.

2. The sum of the coefficients of the expansion of $(a+b)^n$ is equal to $2^n$. For example, in the expansion above we have

$$1+6+15+20+15+6+1=64=2^6$$

3. The sum of the coefficients of terms in odd positions is equal to the sum of the coefficients of terms in even positions. Each one is $2^{n-1}$; for example, in the expansion $(a+b)^6$ the sum of the coefficients of the 1st, 3rd, 5th and 7th terms is equal to the sum of the coefficients of the 2nd, 4th and 6th terms:

$$1+15+15+1=6+20+6=32=2^5$$

# GEOMETRY

## A. PLANE GEOMETRY

### 137. Geometric Constructions

1. *To draw, through a given point C, a straight line parallel to a given straight line AB.*

Open a pair of compasses and draw an arbitrary circle (Fig. 22), centre $C$, such that it intersects $AB$. Using the

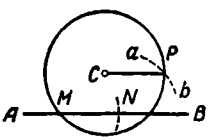

Fig. 22.

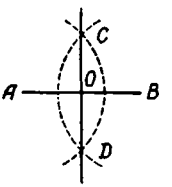

Fig. 23.

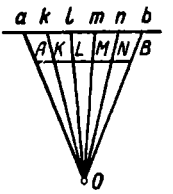

Fig. 24.

same opening of the compasses lay off on $AB$ from one of the points of intersection, say $M$, a line-segment $MN$ (in either direction) and then describe the arc $ab$ from $N$. Joint $C$ to the point $P$ of intersection of the arc $ab$ with the circle. $PC$ is the desired straight line.

2. *To bisect a given line-segment AB.*

Using a pair of compasses with an arbitrary opening (but greater than $^1/_2$ $AB$) describe two arcs from the endpoints of the line $AB$. Join their points of intersection ($C$ and $D$) with a straight line. The intersection point $O$ of the straight lines $AB$ and $CD$ is the midpoint of the line $AB$.

3. *To divide a given line AB into a given number of equal parts.*

In Fig. 24 draw a straight line $ab$ parallel to $AB$; on it lay off as many equal segments of arbitrary length as needed, say $ak = kl = lm = mn = nb$. Draw the straight lines $Aa$, $Bb$ to intersect in the point $O$. Draw straight lines $Ok$, $Ol$, $Om$, $On$. These lines will intersect $AB$ at the points $K$, $L$, $M$, $N$, which divide $AB$ into the required number (in our case, 5) of equal parts.

4. *To divide a given line-segment into parts proportional to given quantities.*

This problem is solved in the same way as Problem 3, but on *ab* we lay off segments proportional to given quantities.

5. *To draw a straight line perpendicular to a given straight line MN from a given point A.*

From an arbitrary point *O* without the given straight line (Fig. 25) draw a circle of radius *OA*. Draw a diameter

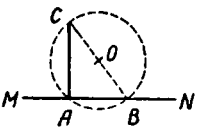

Fig. 25.

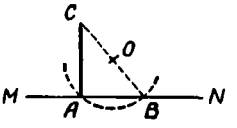

Fig. 26.

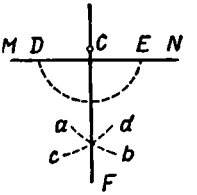

Fig. 27.

*BC* through the second point *B* of intersection of the circle with the line *MN*. Join *A* and the endpoint of the diameter, *C*; *CA* is the desired perpendicular.

This mode of construction is particularly useful when the point *A* lies close to the edge of the paper. The method of solving the next problem (first case) has the same advantage.

6. *To drop a perpendicular from a given point C to a straight line MN.*

From point *C* draw an arbitrary inclined line *CB* (Fig. 26). Find its midpoint *O* (see Problem 2) and from it describe a circle of radius *OB*. The circle also intersects *MN* at point *A*. Join *A* and *C* to get the desired perpendicular.

When *C* lies close to *MN*, this method of construction may result in a considerable error. The following is a preferable alternative construction. From point *C* as centre (Fig. 27), draw an arc *DE* cutting *MN* at points *D* and *E*. From *D* and *E* as centres, draw two arcs *cd*, *ab* of the same radius; they intersect at *F*. Draw *FC* to obtain the desired perpendicular.

7. *For a given vertex K and ray KM, to construct an angle equal to a given angle ABC.*

From the vertex *B* (Fig. 28) describe an arc *PQ* of arbitrary radius. Using the same opening of the compasses,

describe from the centre $K$ an arc $pq$. From point $p$ describe an arc $\alpha\beta$ of radius equal to $PQ$. Join to $K$ the intersection point $q$ of arcs $pq$ and $\alpha\beta$. The angle $qKM$ is the required angle.

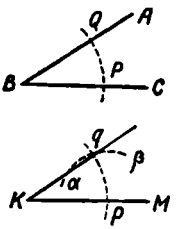


Fig. 28.

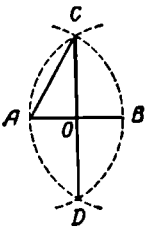


Fig 29.

8. *To construct angles of* 60° *and* 30°.

From endpoints $A$ and $B$ (Fig. 29) of an arbitrary line $AB$ describe two arcs of radius $AB$. Join their intersection points $C$ and $D$ with a straight line which cuts $AB$ at its midpoint $O$. Join $A$ and $C$ with a straight line. $\angle CAO = 60°$, $\angle ACO = 30°$.

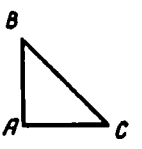


Fig. 30.

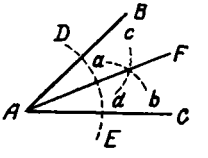


Fig. 31.

9. *To construct an angle of* 45°.

Lay off on the sides of a right angle $BAC$ (Fig. 30) equal segments $AB$ and $AC$ and join their endpoints with a straight line $CB$. The line $BC$ forms 45° angles with $AC$ and $AB$.

10. *To bisect a given angle* $BAC$.

From vertex $A$ (Fig. 31) draw an arc $DE$ of arbitrary radius. From points $D$ and $E$ (where $DE$ cuts arms $AB$ and $AC$) draw the arcs $ab$, $cd$ with arbitrary equal radii (it is most convenient to use the original opening of the compasses). Join their point of intersection to $A$. The resulting straight line $AF$ bisects the angle $BAC$.

11. *To trisect a given angle BAC.*

It is impossible to make this construction with straightedge and compasses alone. Using compasses and a graduated ruler (say with centimetre divisions) the construction can be carried out as follows (Fig. 32): from point *A* describe a

Fig. 32.

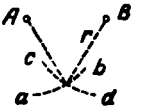


Fig. 33.

circle of arbitrary radius *AC*. Produce *AC* beyond *A*. Place the ruler so that it passes through *B*, then revolve it about *B* until *ED* (the line-segment between the circle and the straight line *AK*) is equal to the radius *AC*. The angle *EDF* is one third of the angle *BAC*.

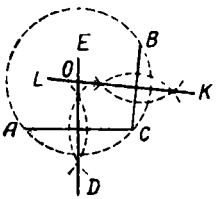


Fig. 34.

12. ***Through two given points A and B to draw a circle of given radius r.***

From points *A* and *B* (Fig. 33) describe arcs *ab* and *cd* of radius *r*. Their point of intersection is the centre of the desired circle.

13. *To draw a circle through three given points A, B, C not lying on a single straight line.*

Referring to Fig. 34, draw straight lines *ED* and *KL* perpendicular to lines *AC* and *BC* at their midpoints (see Problem 2). The point *O* of intersection of these perpendiculars is the centre of the desired circle.

14. *To find the centre of a given arc of a circle.*

Take three points on the arc spaced as far apart as possible. Then proceed as in Problem 13.

15. *To bisect a given arc of a circle.*

Join the ends of the arc by a chord. Draw a perpendicular through the midpoint of the chord (see Problem 2). It bisects the arc (and the chord).

16 *To find the locus from which a given line AB is seen at a given angle α:*

The desired locus (Fig. 35) is in the form of two arcs of equal circles with endpoints at *A* and *B*. (The points *A* and *B* do not belong to the locus). The centres of these arcs are found as follows: draw perpendiculars *AD* and *BK* to the

endpoints of $AB$ (see Problem 5). Construct an angle $KBL = \alpha$. We find point $C$ at the intersection of $BL$ and $AD$. The midpoint $O$ of line $BC$ is the centre of one of the sought-for arcs. The other arc is constructed similarly.

17. *To draw through a given point A a tangent to a given circle.*

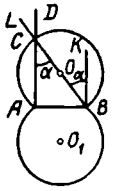

Fig. 35

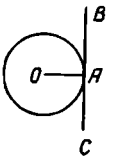

Fig. 36.

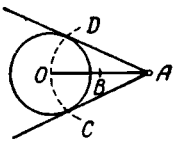

Fig. 37.

If $A$ lies on the circle (Fig. 36), construct $BAC$ perpendicular to radius $OA$ (see Problem 5); $CB$ is the required tangent line.

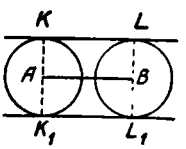

Fig. 38.

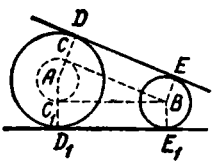

Fig. 39.

If $A$ lies without the circle (Fig. 37), bisect $AO$ (see Problem 2) and from the midpoint $B$ draw arc $CD$ of radius $BO$. Join $D$ and $C$ to $A$ with straight lines. The straight lines $AD$ and $AC$ are the required tangent lines.

18. *To construct an exterior common tangent to two given circles.*

(a) If the radii of the given circles are equal, the problem always has two solutions (Fig. 38). Through centres $A$ and $B$ draw diameters $KK_1$ and $LL_1$ perpendicular to the line $AB$ of centres. Drawing $KL$ and $K_1L_1$ we get the required solutions.

(b) Let the radii of the circles be unequal: $R > r$; from the centre of the larger circle draw a circle of radius $AC = R - r$ (Fig. 39). Draw to this circle from the centre $B$ of the smaller circle the tangent line $BC$ (Problem 17). Join centre $A$ to

the point of contact $C$. Produce it to meet the larger circle at $D$. Draw $BE$ perpendicular to $BC$ to meet the smaller circle at $E$. Join $D$ and $E$. The straight line $DE$ is the desired tangent. The problem admits two solutions ($DE$ and $D_1E_1$) if the smaller circle does not lie wholly within the

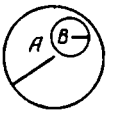


Fig. 40.

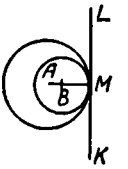


Fig. 41.

larger one. If the smaller circle lies wholly within the larger one (Fig. 40), the problem does not have a solution. In the intermediate case when the circles are tangent internally

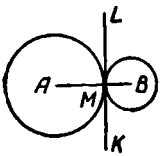


Fig. 42.

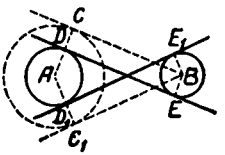


Fig. 43.

(Fig. 41), the problem has one solution: through the point $M$ of internal contact draw $KL \perp AM$.

19. *To construct an interior common tangent to two given circles.*

The problem has no solution if one of the circles lies within the other and also if the circles intersect. In the case of external contact (Fig. 42) the problem has one solution: draw $KL \perp AB$ through $M$.

In all other cases we have two solutions ($DE$ and $D_1E_1$, Fig. 43). From centre $A$ draw a circle of radius equal to the sum of the radii of the given circles. From centre $B$ draw the tangent $BC$ to the constructed circle (Problem 17). Join the point of contact $C$ and the centre $A$; the straight line $AC$ cuts the circle ($A$) at point $D$. From $B$ draw a radius $BE \perp BC$. Join its extremity $E$ to $D$; $ED$ is the desired tangent. The other tangent, $E_1D_1$, is constructed analogously.

20. *To circumscribe a circle about a given triangle ABC.* Draw a circle through vertices $A$, $B$, $C$ (see Problem 13).

21. *To inscribe a circle in a given triangle ABC.* In Fig. 44 bisect two angles of the triangle, say $A$ and $C$ (see Problem 10). From the point $O$ of intersection of the bisectors draw $OD \perp AC$ (see Problem 6). Using radius $OD$ describe the required circle.

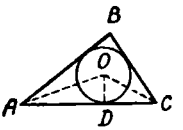


Fig. 44.

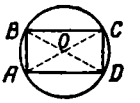


Fig. 45.

22. *To circumscribe a circle about a given rectangle (or square) ABCD.* Draw diagonals $BD$ and $AC$ (Fig. 45). From the point $O$ of their intersection draw a circle of radius $OA$.

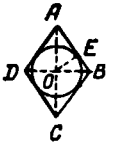


Fig. 46.

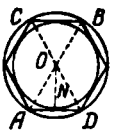


Fig. 47.

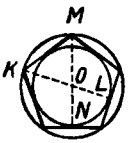


Fig. 48.

It is impossible to circumscribe a circle about an oblique-angled parallelogram.

23. *To inscribe a circle in a rhombus (or square) ABCD.* From the point $O$ of intersection of the diagonals draw $OE \perp AB$ (Fig. 46). The circle with centre $O$ and radius $OE$ is the required circle.

It is not possible to inscribe a circle in a nonequilateral parallelogram.

24. *To circumscribe a circle about a given regular polygon.* If the number of sides is even (Fig. 47), join any two pairs of opposite vertices by straight lines $AB$ and $CD$. From the point $O$ of their intersection describe a circle of radius $OA$.

If the number of sides is odd (Fig. 48), drop perpendiculars $KL$ and $MN$ from vertices $K$ and $M$ onto opposite sides. From the intersection point $O$ describe a circle of radius $OK$.

25. *To inscribe a circle in a given regular polygon.*

The centre of the circle is found as in Problem 24. From the centre drop a perpendicular $ON$ on one of the sides (see Fig. 47). Describe the circle using radius $ON$ (or $OL$, Fig. 48).

26. *To construct a triangle, given three sides a, b, and c.*

Let the longest be $a$. If $a < b+c$, then the desired triangle can be constructed as follows: lay off $BC=a$ (Fig. 49). From its extremities $B$ and $C$ describe arcs $mn$ and $pq$ of radii $c$ and $b$. Join the point $A$ of intersection of the arcs to $B$ and $C$.

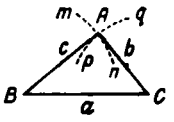

Fig. 49.

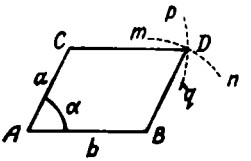

Fig. 50.

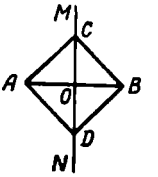

Fig. 51.

If $a > b+c$, then the problem has no solution. In the intermediate case $a=b+c$, only a degenerate triangle meets the conditions: its three vertices lie on one straight line.

27. *To construct a parallelogram, given sides a and b and one angle α.*

Construct $\angle A=\alpha$ (see Problem 7); on its sides lay off $AC=a$, $AB=b$ (Fig. 50). Draw from $B$ an arc $mn$ of radius $a$ and from $C$ an arc $pq$ of radius $b$. Join the point $D$ of intersection of these arcs to $C$ and $B$.

28. *To construct a rectangle, given the base and the altitude.*

Proceed as in Problem 27; construct the right angle $\alpha$ as in Problem 5.

29. *To construct a square, given a side.*

Proceed as in Problems 27 and 28.

30. *To construct a square, given its diagonal AB.*

Through the midpoint of $AB$ (Fig. 51), draw a perpendicular $MN$ to $AB$ (see Problem 2). From $O$, its point of intersection with $AB$, lay off on $MN$ lengths $OC$ and $OD$ equal to $OA$. $ACBD$ is the required square.

31. *To inscribe a square in a given circle.*

Draw two mutually perpendicular diameters $AB$ and $CD$ (Fig. 52). $ACBD$ is the required square.

32. *To circumscribe a square about a given circle.*

Draw two mutually perpendicular diameters $AB$ and $CD$ (Fig. 53). From their extremities as centres, describe four semicircles of radii $OA$. The points of their intersection, $F$, $G$, $H$, and $E$, are the vertices of the required square.

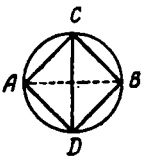

Fig. 52.

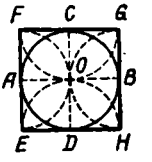

Fig. 53.

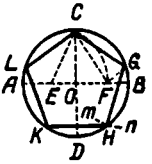

Fig. 54.

33. *To inscribe a regular pentagon in a given circle.*

Draw two mutually perpendicular diameters $AB$ and $CD$ (Fig. 54). Bisect the radius $AO$ to get point $E$. From $E$ draw an arc $CF$ of radius $EC$, cutting the diameter $AB$ at $F$.

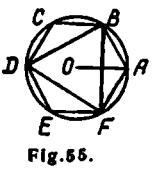

Fig. 55.

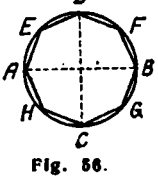

Fig. 56.

From $C$ draw an arc $FG$ of radius $CF$ cutting the given circle at $G$; $CG$ $(=CF)$ is one side of the required figure. With the same radius, draw arc $mn$ from centre $G$ to get one more vertex, $H$, of the desired figure, etc.

34. *To inscribe a regular hexagon and a triangle in a circle.*

Opening the compasses to the radius of the circle, strike arcs on the circumference at points $A$, $B$, $C$, $D$, $E$, $F$ (Fig. 55). Join $A$, $B$, $C$, $D$, $E$, $F$ in succession to obtain a regular hexagon. Joining alternate points, we get a regular (equilateral) triangle.

35. *To inscribe a regular octagon in a given circle.*

Draw two mutually perpendicular diameters $AB$ and $CD$ (Fig. 56). Bisect arcs $AD$, $DB$, $BC$, $CA$ by points $E$, $F$, $G$, $H$ (Problem 15). Join in succession the eight points thus obtained.

36. *To inscribe a regular decagon in a given circle.*

Construct a point $F$ (see Fig. 54) as in Problem 33. $OF$ is a side of the required figure. Opening the compasses to the length $OF$, strike 10 consecutive arcs on the circumference. These are the vertices of the desired figure.

Regular polygons inscribed in a circle and having 7 and 9 sides are not constructible exactly by means of straightedge and compasses.

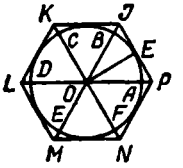


Fig. 57.

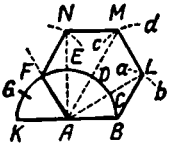


Fig. 58.

37. *To circumscribe about a circle a regular triangle, pentagon, hexagon, octagon, and decagon.*

Mark on the circumference of a circle (Fig. 57) the vertices $A, B, \ldots, F$ of a regular inscribed polygon with the same number of sides (see Problems 33 to 36). Draw radii $OA, OB, \ldots, OF$ and produce them. Bisect the arc $AB$ by the point $E$ (see Problem 15). Through $E$ draw $JP \perp OE$. The length $JP$ between the extensions of adjacent radii is a side of the required figure. On the extensions of the other radii lay off lengths $OK, OL, \ldots, ON$ equal to $OP$. Join the points $J, K, L, \ldots, N, P$ in succession. The polygon $JKLM \ldots NP$ is the required one.

38. *To construct a regular n-gon, given a side a.*

On a length $BK$ (Fig. 58), equal to $2a$, as diameter, construct a semicircle. Divide the semicircle into $n$ equal parts by the points $C, D, E, F, G$ (these are the vertices of a regular inscribed $n$-gon; $n=6$ in our figure). Connect centre $A$ with all the points obtained, except the last two ($K$ and $G$). From $B$ draw an arc $ab$ of radius $AB$, marking point $L$ on the ray $AC$. Using the same radius, from point $L$ draw an arc $cd$, marking point $M$ on the ray $AD$, etc. Join the points $B, L, M, N$ and so on in succession. The polygon $ABLMNF$ is the required one.

The problem is not always solvable by straightedge and compasses; for instance, for $n=7$ and $n=9$ the figure is not constructible since it is impossible, using straightedge and compasses, to divide a semicircle into 7 or 9 equal parts.

### 138. The Subject of Geometry

Geometry (see Sec. 139 on the origin of the word) studies the spatial properties of objects disregarding all other features. For instance, a rubber ball 25 cm in diameter differs from a cast-iron ball of the same diameter in weight, colour, hardness, etc. Geometry disregards all these qualities of the balls and states that their spatial properties (shape and dimensions) are the same. From the viewpoint of geometry, both objects represent a *sphere* of diameter 25 cm.

An object conceptually stripped of all properties except its spatial qualities is called a *geometric body* (solid). A sphere is a geometric solid.

Extending the process of abstraction, we get the concepts of a geometric *surface*, a geometric *line* and a geometric *point*. We mentally separate the surface of a solid from the body it belongs to and divest it of thickness. We deprive a line of thickness and width. A point is without any dimensions whatever. We conceive of a point as serving as the boundary of a line (or of a part of it), a line, as a boundary of a surface, and a surface, as a boundary of a solid. We visualize the point as moving and, in this motion, as generating a line; a line in motion generates a surface, a moving surface generates a solid.

There are no points in nature devoid of dimensions, but there are objects so small that they can conveniently be taken for geometric points. Neither are there any geometric lines or geometric surfaces in nature, but all the properties of lines and surfaces revealed in geometry find multifarious applications in science and engineering. This is due to the fact that geometric concepts are products of the spatial properties of the real world about us. It is the abstract form of geometric concepts which serves to strip these properties of all inessentials and exhibit them in pure form.

### 139. Historical Survey of the Development of Geometry

The notions of geometry originated in remote antiquity. They arose out of the necessity to determine the volumes of various objects (vessels, granaries and the like). The most ancient written records which contain rules for determining

areas and volumes were compiled in Egypt and Babylonia about 4000 years ago. About 2500 years ago the Greeks took over from the Egyptians and Babylonians their geometric findings. This knowledge was first used mainly in measuring land areas, whence the Greek term "geometry", which means "earth measurement".

The Greek scholars discovered numerous geometric properties and set up a harmonious system of geometric knowledge. For the basic starting principles they took the most elementary geometric properties suggested by experience. The other properties were derived from these by logical reasoning.

About 300 B.C. this system took on a finished form as the "Elements" of Euclid, a work which also includes the essentials of theoretical arithmetic. The geometric sections of the "Elements" hardly differ either in content or rigour from present-day school textbooks of geometry.

But we find nothing there about volume, the surface of a sphere, or the ratio of the circumference of a circle to the diameter (although there is a theorem stating that the areas of circles are in the ratio of the squares of the diameters). The approximate value of this ratio was known from experience long before Euclid's time, but it was only in the middle of the third century B.C. that Archimedes (287—212) gave a rigorous proof that the ratio of the circumference of a circle to the diameter (our number $\pi$) lies between $3\frac{1}{7}$ and $3\frac{10}{71}$. Archimedes also proved that the volume of a sphere is less than the volume of a circumscribed cylinder by exactly $1\frac{1}{2}$ times and that the surface of a sphere is $1\frac{1}{2}$ times less than the total surface of a circumscribed cylinder.

The methods used by Archimedes in the solution of the problems mentioned above contain the embryo of the methods of higher mathematics. Archimedes applied these methods to the solution of many difficult problems in geometry and mechanics that were very important to construction work and navigation. For instance, he determined the volumes and the centres of gravity of many solids and studied the question of the equilibrium of floating bodies of diverse shapes.

The Greek geometers investigated the properties of many curves that have theoretical and practical importance. They made a particularly deep study of *conic sections* (see Sec. 167).

In the second century B.C., Apollonius enriched the theory of conic sections with many important discoveries that have remained unsurpassed for eighteen centuries.

In his study of conic sections Apollonius made use of the method of coordinates (see Sec. 211). This method was applied to curves in the plane only in the 1630's by the French mathematicians Fermat (1601—1655) and Descartes (1596—1650). The engineering work of that day did not demand anything beyond plane curves. It was only one hundred years later, when the demands of developing astronomy, geodesy and mechanics became urgent, that the coordinate method was applied to the study of curved surfaces and lines drawn on curved surfaces.

A systematic development of the coordinate method in space was given in 1748 by the great Euler.

For over two thousand years the system of Euclidean geometry was regarded as immutable. But in 1826 the celebrated Russian mathematician N. I. Lobachevsky created a new geometric system. The starting propositions of this system differ from those of Euclid in one point only, yet this difference leads to a multitude of very essential peculiarities. (In Euclidean geometry, through a point $A$ there can pass only one straight line in the same plane as a given straight line $BC$ without intersecting it. In the geometry of Lobachevsky there are infinitely many such straight lines).

Thus, in Lobachevskian geometry the sum of the angles of a triangle is always less than 180° (in Euclidean geometry it is equal to 180°), and the defect (the amount less than 180°) is the greater, the larger the area of the triangle. It might appear that practical experience refutes this and other conclusions made by Lobachevsky. But this is not so. Measuring the angles of a triangle directly, we find their sum to be approximately 180°. The exact value eludes us because of the imperfections of our measuring instruments. On the other hand, all the triangles that we can measure are too small for us to notice the defect in the sum of the angles **via direct measurements.**

Subsequent development of Lobachevsky's ideas showed that the Euclidean system is insufficient for a study of many problems of astronomy and physics where we have to do with bodies of immense proportions. However, it suffices fully to handle all practical problems of our mundane life. And since, besides, it has the advantage of simplicity, it will continue to be used in engineering calculations and it will continue to be studied at school.

### 140. Theorems, Axioms, Definitions

An argument that establishes some property is called a *proof* or demonstration. The property demonstrated is called a *theorem*. In the proof of geometric theorems we proceed from earlier established properties, some of which in turn are theorems; however some are considered in geometry to be basic and are assumed to be true without proof. Such properties, or self-evident truths, are called *axioms*.

Axioms originated from experience, and it is experience again that verifies the truth of axioms as a system. This verification consists in the fact that all the theorems of geometry are in agreement with experiment, which would not be the case if the system of axioms were false.

Not a single geometric property taken separately is an axiom, since it can always be demonstrated on the basis of other properties. For instance, the geometric axiom on the property of parallel lines reads: "only one line can be drawn parallel to a given line through a given point not on this line" (the axiom of parallels). Proof is given, on the basis of this axiom (and a number of others), that the "sum of angles of a triangle is 180°". Yet we could take this latter property for the axiom in place of the axiom of parallels (leaving the other axioms unchanged). Then the property of parallel straight lines may be demonstrated thus becoming a theorem.

Thus, a system of axioms can be chosen in a variety of ways. The sole requirement is that the set of axioms be sufficient to derive all the other geometric properties. In geometry, the attempt is made to reduce the number of axioms as far as possible. This is done in order to elucidate the logical relationships between the separate properties.

Axioms are preferably chosen from among the most elementary geometric properties. True, opinions may differ as to the simplicity of a property.

Some of the concepts of geometry are taken to be initial and their content is extracted from experience alone (such, for instance, is the notion of a point). All other concepts are derived from these initial ones. They are defined and such explanations are called *definitions*. Every geometric definition either stems directly from the initial concepts or is based on earlier defined concepts.

One and the same geometric concept may be defined differently. For example, the diameter of a circle may be defined as a chord that passes through the centre, or as the chord of greatest length. We can take one of these properties as

the definition and then prove the other property. Preference is given to the most elementary property; here too, incidentally, it is impossible to ensure complete and general agreement.

### 141. Straight Line, Ray, Line-Segment

A straight line can be mentally extended in either direction indefinitely. A straight line that is bounded on one side is called a *half-line*, or a *ray*. A straight line bounded on both sides is called a *line-segment*.

### 142. Angles

An angle (symbol: $\angle$) is a figure (Fig. 59) formed by two rays, $OA$ and $OB$ (the *arms* or *sides* of the angle), emanating from a single point $O$ (*vertex* of the angle).

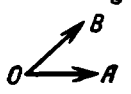

Fig. 59.

An angle is measured by the amount of rotation about the vertex $O$ which carries ray $OA$ to $OB$. Wide use is made of two systems of measuring angles: in terms of *radians* and *degrees*. They differ in the choice of the unit of measurement. Radian measure is discussed in Sec. 180.

*Measurement of angles in degrees.* * The unit here is one degree = 1/360 of one complete rotation (symbol: °). Thus, one complete rotation (exemplified by the hand of a clock moving from 0 hrs to 12 hrs) constitutes 360°. A degree consists of 60 minutes (denoted'); a minute consists of 60 seconds (″). For example, 42°33′21″ is read as "42 degrees, 33 minutes, 21 seconds".

An angle of 90° (i.e., one fourth of a complete revolution) is called a right angle (Fig. 60) and is denoted by the letter $d$.

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* The degree unit of angular measure goes back to remote antiquity (see Sec. 21, Item 4). During the first French bourgeois revolution of 1793, a centesimal system of angular measurement was introduced in keeping with the decimal (metric) system of measures which was introduced at that time. In that system a right angle was divided into 100 degrees (called grades), one degree into 100 minutes, and one minute into 100 seconds. This system is still used today but not widely, mostly in geodetic measurements.

An angle less than 90° is called *acute* (*AOB* in Fig. 59), an angle greater than 90° is called *obtuse* (Fig. 61). The straight lines that form a right angle are called *perpendicular lines*.

Fig. 60.

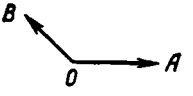


Fig. 61.

Fig. 62.

*Signs of angles.* It is often important to know in what direction a ray is being rotated. In angular measurement, it is ordinarily taken that counterclockwise rotation of a ray corresponds to *positive* values, while clockwise rotation indicates *negative* values. For example, if the ray *OA* moves to

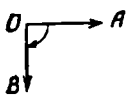
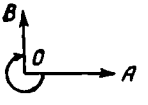


Fig. 63.



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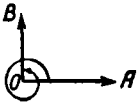

Fig. 65.

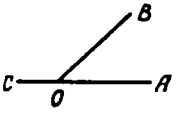

Fig. 66.

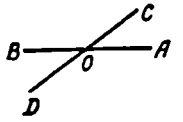

Fig. 67.

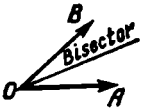

Fig. 68.

coincidence with *OB*, as indicated in Fig. 62, then $\angle AOB = +90°$. In Fig. 63, $\angle AOB = -90°$. In Fig. 64, $\angle AOB = -270°$. One and the same arrangement of rays may correspond to different angular measures depending on the type of rotation. Thus, $\angle AOB$ in Fig. 65 may be taken equal to $+450°$. In elementary geometry, angular measurements are always taken to be positive and are considered to measure the smallest rotation so that one does not measure angles greater than 180°.

*Adjacent angles.* In Fig. 66 the pair of angles *AOB* and *COB* with common vertex *O* and common side *OB* are adjacent angles. The two other sides, *OA* and *OC* are extensions

of one another. The sum of these two adjacent angles is 180° (2*d*). In Fig. 70 we have adjacent angles *AON* and *NOB* which are less than 180°. Angles which have a common vertex, one common arm and are on opposite sides of the common arm, are called adjacent angles.

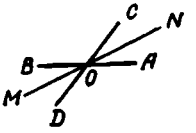


Fig. 69.

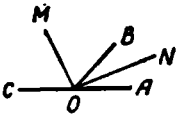


Fig. 70.

*Vertically opposite angles* (or simply, vertical angles) are those which have a common vertex and the sides of one are the extended sides of the other. In Fig. 67, $\angle AOC$ and $\angle DOB$ (also $\angle COB$ and $\angle AOD$) are vertical angles. Vertical angles are equal ($\angle AOC = \angle BOD$).

When speaking of "*an angle between two straight lines*", we mean any one of the four angles produced (usually the acute angle).

The *bisector* of an angle is the line that divides the angle into two equal parts (Fig. 68). The bisectors of vertical angles (*OM* and *ON* in Fig. 69) form a single straight line. The bisectors of adjacent supplementary angles are perpendicular to each other (Fig. 70).

### 143. Polygons

A plane figure formed by a closed series of rectilinear segments is called a *polygon*. Figure 71 depicts a hexagon *ABCDEF*. The points *A*, *B*, *C*, *D*, *E*, *F*, are the *vertices of the polygon*; the angles at these points (the angles of the polygon) are denoted by $\angle A$, $\angle B$, $\angle C$, ..., $\angle F$. The lines *AC*, *AD*, *BE*, etc. are *diagonals*; *AB*, *BC*, *CD*, etc. are the *sides of the polygon*. The sum of the lengths of the sides, $AB + BC + CE + \ldots + FA$, is termed the *perimeter* and is denoted by *p*, sometimes $2p$ (then *p* is the semiperimeter).

Only *simple* polygons (that is, such that the contour has no self-intersections) are studied in elementary geometry. Polygons whose contours have self-intersections are called *star polygons*. A star polygon *ABCDE* is shown in Fig. 72.

If all the diagonals of a polygon lie inside the figure, the polygon is termed *convex*. The hexagon shown in Fig. 71 is convex; the pentagon in Fig. 73 is nonconvex, or concave (the diagonal $EC$ lies without the polygon).

The sum of the interior angles in any convex polygon is equal to $180^\circ\ (n-2)$, where $n$ is the number of sides of the polygon. *

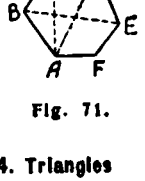

Fig. 71.

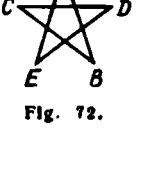

Fig. 72.

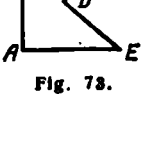

Fig. 73.

### 144. Triangles

A *triangle* (symbols: $\triangle$, plural $\triangle$s) is a polygon having three sides. The sides of a triangle are frequently denoted by lower-case letters corresponding to the labels of the opposite vertices. If all three angles of a triangle are acute, the figure is called an *acute-angle triangle* (Fig. 74); if one of the angles is a right angle, it is called a *right triangle* (Fig. 75); the sides forming the right angle are called *legs* ($a$, $b$); the side opposite the right angle is called the *hypotenuse* ($c$). If one of the angles is obtuse (say, $\angle B$ in Fig. 76), then we have an *obtuse-angle triangle*.

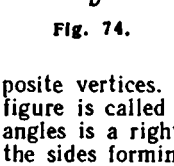

Fig. 74.

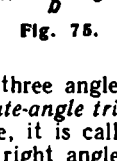

Fig. 75.

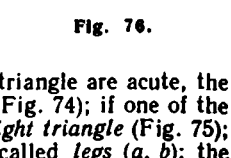

Fig. 76.

* In geometry textbooks this property is ordinarily stated only for convex polygons, yet it is valid for all "simple" polygons. Note that in a concave polygon, one or several interior angles exceed 180°. Thus, in the concave pentagon shown in Fig. 73, two angles are right angles, two are 45° angles, and one contains 270°. The sum of the angles is $180^\circ\ (5-2)=540^\circ$.

Triangle $ABC$ (Fig. 77) is an *isosceles* triangle: two of its sides are equal ($b=c$). When three sides are equal ($a=b=c$), as in Fig. 78, we have an *equilateral triangle.*

In any triangle, the greater angle is opposite the greater side; if the sides are equal, the opposite angles are equal;

Fig. 77.

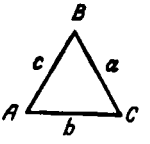

Fig. 78.

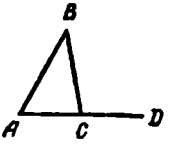

Fig. 79.

and conversely, if the angles are equal, the opposite sides are equal. In particular, equilateral triangles are also equiangular, and conversely.

*The sum of the angles of any triangle is* 180°. Each angle in an equilateral triangle is equal to 60°.

Producing one of the sides of a triangle ($AC$ in Fig. 79), we get an *exterior angle*, $\angle BCD$. *An exterior angle is equal to the sum of the nonadjacent interior angles:* $\angle BCD = \angle A + \angle B$.

Any side of a triangle is less than the sum and more than the difference of the other two sides ($a < b+c$; $a > b-c$).

*The area of a triangle is equal to one-half the product of the base and the altitude* (see Sec. 146 on the altitude of a triangle): $S = 1/2\, ah_a$.

### 145. Congruence of Triangles

Two triangles are congruent if the following elements are respectively equal:

(1) two sides and the included angle; for example, $AB = A'B'$; $AC = A'C'$; $\angle A = \angle A'$ (Fig. 80);

(2) two angles and the adjacent side; for example, $\angle A = \angle A'$; $\angle C = \angle C'$; $AC = A'C'$;

(2a) two angles and the side opposite one of them, for example, $\angle A = \angle A'$; $\angle B = \angle B'$; $AC = A'C'$;

Fig. 80.

(3) three sides: $AB = A'B'$; $BC = B'C'$; $AC = A'C'$;

(4) two sides and the angle opposite the greater side; for example, $AB = A'B'$, $BC = B'C'$, $\angle A = \angle A'$ in Fig. 80, where $BC$ is greater than $AB$. If equal angles lie opposite

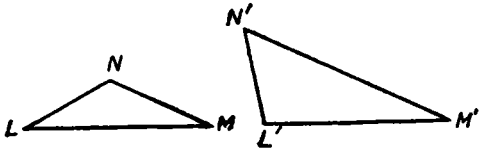


Fig. 81.

smaller sides, then the triangles may not be congruent. For example, the triangles $LMN$ and $L'M'N'$ in Fig. 81 are not congruent although $LM = L'M$ and $LN = L'N'$ and $\angle M > \angle M'$. Here, the angles $M$, $M'$ lie opposite the smaller sides $LN$, $L'N'$.

### 146. Remarkable Lines and Points of the Triangle

The *altitude* (or height) of a triangle is the perpendicular drawn from any vertex of the triangle to the opposite side or its extension (the side to which the perpendicular is drawn is then called the *base* of the triangle). In an obtuse-angle

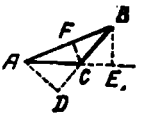


Fig. 82.

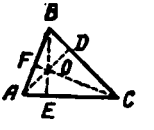


Fig. 83.

tri ngle ($ABC$, in Fig. 82), two altitudes ($AD$, $BE$) fall on the extensions of the sides, outside the triangle, and the third altitude ($CF$) falls inside the triangle. In an acute-angle triangle (Fig. 83), all three altitudes lie within the triangle. In a right triangle, the legs serve as altitudes. The three altitudes of a triangle always meet in one point called the *orthocentre*. In an obtuse-angle triangle the orthocentre lies

outside the triangle, in a right triangle it coincides with the vertex of the right angle.

The altitude of a triangle dropped onto side $a$ is denoted by $h_a$. It is expressed in terms of the three sides by the following formula:

$$h_a = \frac{2\sqrt{p(p-a)(p-b)(p-c)}}{a}$$

where

$$p = \frac{a+b+c}{2}$$

*Median.* A straight line which joins a vertex of a triangle to the midpoint of the opposite side is called a median. The three medians of a triangle ($AD$, $BE$, $CF$ in Fig. 84) are

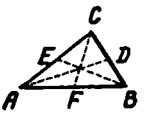


Fig. 84.

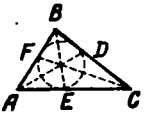


Fig. 85.

concurrent (that is, intersect in one point: the point is always inside the triangle). This point is the centroid (the centre of gravity of the triangle). It divides each median in the ratio 2:1 (reckoning from the vertex). The median which joins the vertex $A$ of a triangle to the midpoint of side $a$ is denoted by $m_a$. It is given, in terms of the sides of the triangle, by the formula

$$m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$$

An *angle bisector* of a triangle (see Sec. 142) is the line segment from the given vertex to its intersection with the opposite side. The three angle bisectors of a triangle ($AD$, $BE$, $CF$, in Fig. 85) intersect in one point (always inside the triangle) which is the centre of the inscribed circle (see Sec. 156). The bisector of angle $A$ is denoted by $\beta_a$. Its length is given, in terms of the sides of the triangle, by the following formula:

$$\beta_a = \frac{2}{b+c}\sqrt{bcp(p-a)}$$

where $p$ is the semiperimeter. A bisector divides the opposite

side into parts proportional to the sides adjacent to it. In Fig. 85, $AE:EC=AB:BC$.

**Example.** $AB=30$ cm, $BC=40$ cm, $AC=49$ cm. Find $AE$ and $EC$. The two parts ($AE$ and $EC$) into which $AC=49$ cm is to be divided are in the ratio 30:40 or 3:4. Taking $x$ as the scale unit ($x=\frac{1}{3}AE$, $x=\frac{1}{4}EC$), we have $AC=3x+4x=7x$, $x=AC:7=49:7=7$, whence $AE=3x=21$; $EC=4x=28$.

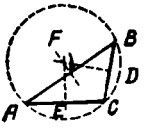


Fig. 86.

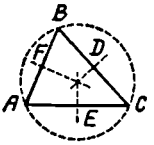


Fig. 87

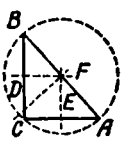


Fig. 88.

The three perpendicular lines drawn to the sides of a triangle through their midpoints ($D$, $E$, $F$, in Figs. 86, 87, 88) meet at one point which is the circumcentre (see Sec. 156). In an obtuse-angle triangle (Fig. 86) this point lies outside the triangle, in an acute-angle triangle (Fig. 87) it lies inside the triangle, in a right triangle it lies at the midpoint of the hypotenuse (Fig. 88).

In an isosceles triangle, the altitude, the median, the angle bisector and the perpendicular drawn to the midpoint of the base (which in an isosceles triangle is always the side different from the other two) all coincide. The same situation holds true for all three sides of an equilateral triangle. In all other cases none of these lines coincides with any other one. The orthocentre, the centroid, the centre of the inscribed circle and the circumcentre coincide only in an equilateral triangle.

### 147. Orthogonal Projections. Relationships Between the Sides of a Triangle

The *orthogonal projection* (or, briefly, *projection*) of a point on a straight line is the foot of the perpendicular from the point to the line. In Fig. 89, the points $a$, $b$, $c$, $d$ are projections of the points $A$, $B$, $C$, $D$ on the straight line $MN$. The projection of $AB$ on $MN$ is the line segment $ab$ of the

straight line $MN$ bounded by the projections $a$ and $b$ of the endpoints of $AB$. The line segment $bc$ is the projection of $BC$, etc. This is denoted as $ab = \text{pr}_{MN} AB$ or, briefly, $ab = \text{pr}\, AB$.

The sum of the projections of the segments of a polygonal line is equal to the projection of the closing segment. In Fig. 89, $\text{pr}\, AD = \text{pr}\, AB + \text{pr}\, BC + \text{pr}\, CD$. For complete generality, we must regard the projection of a line as an

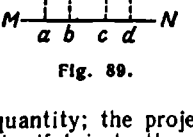


Fig. 89.

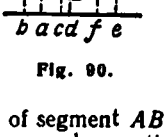


Fig. 90.

algebraic quantity; the projection $ab$ of segment $AB$ is taken to be *positive* if $b$ is to the right of $a$, and *negative* if $b$ is to the left of $a$. Thus, in Fig. 90, $\text{pr}\, AB = ab$ is negative; $\text{pr}\, BC = bc$, $\text{pr}\, CD = cd$, $\text{pr}\, DE = de$ are positive; $\text{pr}\, EF = ef$ is negative. Therefore, the (algebraic) sum of the projections of the segments of the polygonal line $ABCDEF$ is obtained

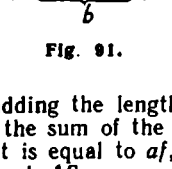


Fig. 91.

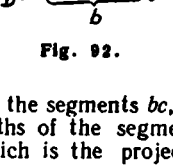


Fig. 92.

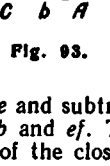


Fig. 93.

by adding the lengths of the segments $bc$, $cd$, $de$ and subtracting the sum of the lengths of the segments $ab$ and $ef$. The result is equal to $af$, which is the projection of the closing segment $AF$.

The square of a side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of one of the two sides by the projection, on it, of the other. Using the notations of Figs. 91 and 92, we have

$$a^2 = b^2 + c^2 - 2b \cdot \text{pr}_{AC} AB \qquad (1)$$

If $x$ denotes the length of a projection (a positive number), then, when angle $A$ is acute ($\text{pr}_{AC} AB = x$ in Fig. 91),

$$a^2 = b^2 + c^2 - 2bx \qquad (2)$$

and when angle $A$ is obtuse ($\text{pr}_{AC}AB = -x$ in Fig. 92),

$$a^2 = b^2 + c^2 + 2bx \tag{3}$$

If angle $A$ is a right angle (Fig. 93), then $\text{pr}_{AC}AB = 0$ and we have

$$a^2 = b^2 + c^2 \tag{4}$$

the square of the hypotenuse is equal to the sum of the squares of the legs (this is known as the Pythagorean theorem *). The theorem of Pythagoras finds extensive applications in both practical and theoretical situations.

Formula (1) can also be written as

$$a^2 = b^2 + c^2 - 2bc\cos A$$

(see Sec. 199).

### 148. Parallel Straight Lines

Two straight lines $AB$ and $CD$ (Fig. 94) are called *parallel lines* if they lie in one plane and do not meet however far produced. Symbol: $AB \parallel CD$. The distance between the lines is everywhere the same.

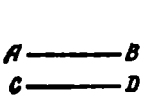


Fig. 94.

Fig. 95.

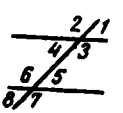


Fig. 96.

All straight lines parallel to the straight line $AB$ are parallel among themselves.

It is considered that two parallel straight lines form an angle equal to zero (there is no angle at all in the direct sense of the word).

If two rays belong to parallel straight lines, then the angle between the rays is taken to be zero when the direction of the rays is the same, and 180° when the directions of the rays are opposite.

Straight lines ($AB$, $CD$, $EF$, Fig. 95) perpendicular to a single straight line $MN$ are themselves parallel. Conversely,

* The theorem is credited to Pythagoras, a Greek philosopher who lived in 6th and 5th centuries B. C. Actually this theorem was known in the Ancient East 20 centuries before the Christian Era.

the straight line $MN$ perpendicular to one of the parallel lines is perpendicular to all others. All lines perpendicular to one of two parallel straight lines are perpendicular to the other. The lengths of these perpendiculars between the two parallel lines are equal. Their common length is the *distance* between the parallel straight lines.

A straight line (called a transversal) that cuts two parallel straight lines forms eight angles (Fig. 96), pairs of which have the following names: (1) corresponding angles (*1* and *5*; *2* and *6*; *3* and *7*; *4* and *8*); these angles are pairwise equal: $\angle 1 = \angle 5$; $\angle 2 = \angle 6$; $\angle 3 = \angle 7$; $\angle 4 = \angle 8$;

(2) alternate interior angles (*4* and *5*; *3* and *6*); they are pairwise equal;

(3) alternate exterior angles (*1* and *8*; *2* and *7*), they are also pairwise equal;

(4) interior angles on the same side of the transversal (*3* and *5*; *4* and *6*); the sum of these angles is equal to 180° ($\angle 3 + \angle 5 = 180°$; $\angle 4 + \angle 6 = 180°$);

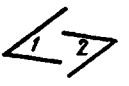


Fig. 97.

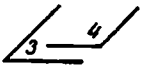


Fig. 98.

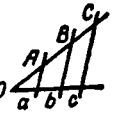


Fig. 99.

(5) exterior angles on the same side of the transversal (*1* and *7*; *2* and *8*); the sum of these angles is equal to 180° ($\angle 1 + \angle 7 = 180°$; $\angle 2 + \angle 8 = 180°$). *

Angles *with corresponding sides parallel* are either equal (if both are acute or both are obtuse) or their sum is equal to 180°; in Fig. 97 $\angle 1 = \angle 2$; in Fig. 98, $\angle 3 + \angle 4 = 180°$. Angles with corresponding sides perpendicular are likewise either equal or constitute a total of 180°.

Parallel straight lines that intersect the sides of an angle, as shown in Fig. 99, intercept proportional lengths on the sides of the angle:

$$\frac{OA}{Oa} = \frac{OB}{Ob} = \frac{OC}{Oc} = \frac{AB}{ab} = \frac{BC}{bc} = \frac{AC}{ac}, \text{ etc.}$$

* When two nonparallel straight lines are cut by a transversal the angles formed bear the same names as those given above, but the relationships between the angles no longer hold true.

### 149. The Parallelogram and the Trapezoid

A *parallelogram* ($ABCD$ in Fig. 100) is a quadrilateral in which both pairs of opposite sides are parallel. The opposite sides of a parallelogram are equal: $AB=CD$, $AD=BC$. Any two opposite sides may be taken as *bases*. The perpendicular distance between them is called the *altitude* ($BF$). The diagonals of a parallelogram bisect each other ($AO=OC$, $BO=OD$). The opposite angles of a parallelogram are equal ($\angle A=\angle C$, $\angle B=\angle D$). *The sum of the squares of the diagonals is equal to the sum of the squares of the four sides:* $AC^2+BD^2=AB^2+BC^2+CD^2+AD^2=2(AB^2+BC^2)$. The area $S$ of a parallelogram is equal to the product of the base ($a$) by the altitude ($h_a$):

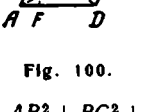

Fig. 100.

$$S=ah_a$$

*Distinguishing features of parallelograms.* A quadrilateral $ABCD$ is a parallelogram provided that:

(1) the opposite sides are equal ($AB=CD$, $BC=DA$);

(2) two opposite sides are equal and parallel

($AB=CD$, $AB \parallel CD$);

(3) the diagonals bisect each other;

(4) opposite angles are equal ($\angle A=\angle C$, $\angle B=\angle D$).

If one of the angles of a parallelogram is a right angle, then all angles are right angles. Such a parallelogram is

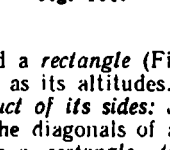

Fig. 101.

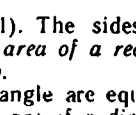

Fig. 102.

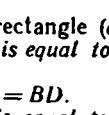

Fig. 103.

called a *rectangle* (Fig. 101). The sides of a rectangle ($a$, $b$) serve as its altitudes. *The area of a rectangle is equal to the product of its sides:* $S=ab$.

The diagonals of a rectangle are equal: $AC=BD$.

*In a rectangle, the square of a diagonal is equal to the sum of the squares of the sides:* $AC^2=AD^2+DC^2$.

If a parallelogram has all sides equal, it is called a *rhombus* (Fig. 102).

In a rhombus, the diagonals are mutually perpendicular ($AC \perp BD$) and bisect the angles ($\angle DCA = \angle BCA$, etc.).

*The area of a rhombus is equal to half the product of the diagonals:*

$$S = \frac{1}{2} d_1 \cdot d_2$$

where $d_1 = AC$, $d_2 = BD$.

A *square* is a parallelogram with right angles and equal sides (Fig. 103). A square is a particular type of rectangle and also a particular type of rhombus. It therefore has all their properties, as given above.

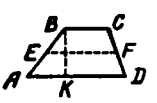


Fig. 104.

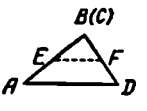


Fig. 105.

Fig. 106.

A *trapezoid* is a quadrilateral having one pair of opposite sides parallel ($BC \parallel AD$, Fig. 104). A parallelogram may be considered as a particular type of trapezoid.

The parallel sides are called the *bases* of the trapezoid, the other two sides ($AB$, $CD$) are called its *nonparallel sides*. The perpendicular distance ($BK$) between the bases is called the *altitude* of the trapezoid. The line $EF$ joining the midpoints of the nonparallel sides is called the *median* (or *midline*) of the trapezoid.

*The median of a trapezoid is equal to one-half the sum of the bases:* $EF = \frac{1}{2}(AD + BC)$, *and is parallel to them:* $EF \parallel AD$.

*The area of a trapezoid is equal to the product of the median by the altitude:*

$$S = \frac{1}{2}(a + b)h$$

where $a = AD$, $b = BC$, $h = BK$.

A triangle is the limiting (degenerate) case of a trapezoid when one of the bases shrinks to a point (Fig. 105). A degenerate trapezoid preserves its properties; for example, the line joining the midpoints $E$ and $F$ of the sides of triangle $ABD$

(the median of the triangle) is parallel to side $AD$ and is equal to one-half of it.

A trapezoid with equal nonparallel sides is called an *isosceles trapezoid* ($AB = CD$ in Fig. 106). The base angles of an isosceles trapezoid are equal ($\angle A = \angle D$, $\angle B = \angle C$).

### 150. Similarity of Plane Figures. Similar Triangles

If all the dimensions of a plane figure are changed (increased or decreased) in the same ratio (the *ratio of similitude*), then the old and new figures are called *similar figures*. For instance, a picture and a photograph of the picture.

*In two similar figures, any corresponding angles are equal;* that is, if points $A$, $B$, $C$, $D$ in one figure correspond to points $a$, $b$, $c$, $d$ of another, then $\angle ABC = \angle abc$, $\angle BCD = \angle bcd$, etc.

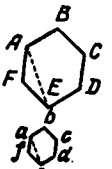

Fig. 107.

Two polygons ($ABCDEF$ and $abcdef$ in Fig. 107) are similar if they have equal angles ($\angle A = \angle a$, $\angle B = \angle b$, ..., $\angle F = \angle f$) and their corresponding sides are proportional $\left(\frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \ldots = \frac{FA}{fa}\right)$. This ensures the proportionality of all other corresponding elements of the polygons; for example, the diagonals $AE$ and $ae$ have the same ratio as the sides $\left(\frac{AE}{ae} = \frac{AB}{ab}\right)$. However, the proportionality of the sides of polygons does not suffice to make them similar; for example, in Fig. 108 the sides of the quadrilateral $ABCD$ (square) are proportional to the sides of the quadrilateral $abcd$ (rhombus); each side of the square is twice that of the rhombus. But the diagonals of the square did not diminish in the same proportion (one diminished more than twice, the other, less than twice) because the angles of the rhombus $abcd$ are not equal to the angles of the square $ABCD$.

In the similarity of triangles, on the other hand, the proportionality of the sides is sufficient: *two triangles are similar if their sides are proportional.* Thus, if the sides of a triangle $ABC$ (Fig. 109) are twice the length of the sides of triangle $abc$, then the angle bisector $BD$ is twice the bisector $bd$, the altitude $BE$ is twice the altitude $be$, etc., and the corresponding angles are equal ($\angle A = \angle a$, $\angle B = \angle b$, $\angle C = \angle c$).

*If the corresponding angles of two triangles are equal, the triangles are similar* (it is sufficient to detect the equality of two pairs of angles because the sum of the angles of a triangle is always 180°). This criterion is not sufficient for arbitrary polygons. For example, the square $ABCD$ and the rectangle $abcd$ (Fig. 110) have equal corresponding angles, but the figures are not similar.

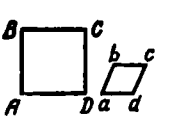


Fig. 108.

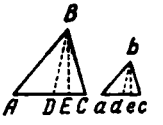


Fig. 109.

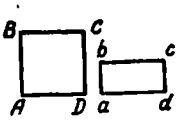


Fig. 110.

*Triangles are also similar when two sides of one are proportional to two sides of the other and the included angles are equal* (that is, if $\frac{AB}{ab}=\frac{BC}{bc}$ and $\angle B=\angle b$).

*Right triangles are similar if the hypotenuse and a leg of one are proportional to the hypotenuse and a leg of the other.*

Any two circles are similar, one being an increased or decreased version of the other.

The areas of similar figures (say, polygons) *are proportional* to the squares of their corresponding lines (sides, for example). In particular, the areas of circles stand in the same proportion to one another as the squares of the radii or the diameters. It would therefore be a serious mistake to consider that the ratio of the areas of two circles is equal to the ratio of their diameters. This mistake is often made, however.

**Example 1.** A circular metal disk of diameter 20 cm weighs 2.4 kg. How much does a disk (of the same material and thickness) 10 cm in diameter weigh?

To reason that the diameter of the small disk is one-half that of the large disk and on that basis to say that the small disk weighs one-half the large disk, or 1.2 kg, would be a grave mistake.

Here is the correct solution. Since the material and thickness of the disk remain the same, the weights are proportional to the areas, and the ratio of the area of the small disk to the area of the large disk is $\left(\frac{10}{20}\right)^2=\frac{1}{4}$. Hence, the weight of the small disk is $2.4\cdot\frac{1}{4}=0.6$ kg.

**Example 2.** The population of a country, call it $A$, is given as 8.2 million, that of some country $B$, 4.1 million. If the population of $B$ is shown diagrammatically as a square with side 10 cm, what will the side of the square depicting the population of $A$ be?

Denoting the required side by $a$, we have

$$\left(\frac{a}{10}\right)^2 = \frac{8.2}{4.1} = 2; \quad \frac{a}{10} = \sqrt{2} \approx 1.4; \quad a \approx 14 \text{ cm}$$

### 151. Loci and Circles

The *locus* (plural, loci) of points having a given property is the totality of the points satisfying the given conditions.

The circle is a locus of points in the plane that are equidistant from one point, the centre.

Equal line-segments joining the centre to points of the circumference are called *radii*, (singular, *radius*, denoted by $r$ or $R$). Part of the circumference (say $AmD$ in Fig. 111)

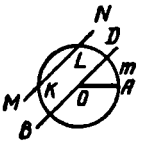


Fig. 111.

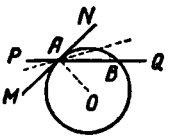


Fig. 112.

is called an *arc*. The straight line $MN$ passing through two points of the circumference is called a *secant*; the portion of it, $KL$, lying inside the circle is a *chord*. A chord increases as it approaches the centre of the circle. The chord $BD$ passing through the centre ($O$) is called the *diameter*, denoted by $d$ or $D$. The diameter is equal to two radii ($d = 2r$).

A *circle* is a plane curve consisting of all points at a given distance (called the radius) from a fixed point in the plane, called the centre. A circle also denotes the region of a plane all points of whose boundary are at a given distance (called the radius) from a fixed point in the plane (called the centre).

*Tangent line.* Let the secant $PQ$ (Fig. 112) pass through the points $A$ and $B$ of the circle. Let the point $B$ move along the circumference approaching $A$. The secant line $PQ$ will change its position as it rotates about $A$. As $B$ appro-

aches $A$, the secant $PQ$ will tend to a certain limiting position $MN$. The straight line $MN$ is called the *tangent* to the circle at the point $A$. The tangent and the circle have only one point in common. * We may consider a tangent to be a degenerate secant.

A tangent to a circle is perpendicular to the radius $OA$ drawn to the point $A$ of contact.

Fig. 113.

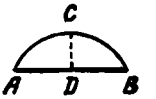


Fig. 114.

From a point without a circle we can draw two tangents to the circle; the tangents will be of equal length (see Fig. 120).

A *segment* of a circle is the area between a chord and the arc subtended by the chord (in Fig. 114 the arc $ACB$ subtended by the chord $AB$).

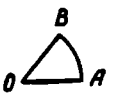


Fig. 115.

Fig. 116.

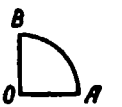


Fig. 117.

The perpendicular drawn from the midpoint of the chord $AB$ to intersection with the arc $AB$ is called the *sagitta* of the arc $AB$. The length of the sagitta $DC$ (Fig. 114) is the *altitude* of the segment.

A *sector* of a circle is that portion of the circle bounded by two radii of the circle and one of the arcs which they intercept (Figs. 115 and 116). A sector with radii that form an angle of 90° is called a *quadrant* (Fig. 117).

* This property is ordinarily taken as a definition of a tangent to a circle. For other lines, however, this definition may prove invalid. For example, $MN$ in Fig. 113 is tangent to the line $CADE$ at point $A$, but $MN$ has, besides $A$, a point $N$ that belongs to $CADE$ as well. The definition we give of a tangent as the limiting position of a secant is applicable to all lines.

### 152. Angles in a Circle. The Length of the Circumference and of an Arc

A *central angle* is formed by two radii ($\angle AOB$ in Fig. 118).

An *inscribed angle* is formed by two chords ($CA$ and $CB$ in Fig. 119) issuing from a common point of the circle ($\angle ACB$ in Fig. 119).

A *circumscribed angle* is one formed by two tangents issuing from one point ($CA$ and $CB$ which form $\angle ACB$ in Fig. 120).

Fig. 118.

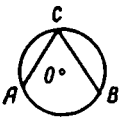


Fig. 119.

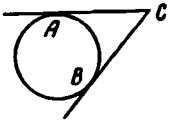


Fig. 120.

The *length of an arc* described by the endpoint of a radius is proportional to the magnitude of the corresponding central angle; for this reason, the arcs of one and the same circle may be measured like angles, by degrees (see Sec. 142). Namely, 1° of arc is taken to be $\frac{1}{360}$ of the circle (that is, an arc whose central angle is equal to 1°). The entire circle contains 360°, one half is 180°.

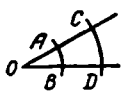


Fig. 121.

To avoid errors that occur frequently, note that the value of the central angle is quite independent of the length of the radius whereas the magnitude of the corresponding arc is proportional to the radius. Thus, in Fig. 121, the central angle preserves the same value irrespective of whether we form it with the radii $OC$ and $OD$ or $OA$ and $OB$, which are half that length. Now the arcs $AB$ and $CD$ are of unequal length although they have the same number of degrees: arc $AB$ is shorter than arc $CD$.

Generally, the length of an arc is proportional to (1) the radius and (2) the magnitude of the corresponding central angle.

The length of the circumference $p$ constitutes about $3\frac{1}{7}$ the length of the diameter: $p \approx 3\frac{1}{7}d$. In other words, the

ratio of the lengths of circumference and diameter is approximately $3\frac{1}{7}$:

$$\frac{p}{d} \approx 3\frac{1}{7}$$

The exact ratio $\frac{p}{d}$ is denoted by the Greek letter pi ($\pi$):

$$\frac{p}{d} = \pi \tag{1}$$

$3\frac{1}{7}$ is an approximate (too high) value of the number $\pi$. The number $\pi$ is irrational (see Sec. 91), which means it cannot be written exactly as a fraction. To five decimal places, pi is given as 3.14159. For practical purposes it suffices to take the value $\pi \approx 3.14$, which is slightly less (the difference is inessential) than $\pi \approx 3\frac{1}{7}$.

Formula (1) yields

$$p = \pi d \tag{2}$$

or

$$p = 2\pi r \tag{3}$$

The length of 1° of arc is

$$p_{1^\circ} = \frac{2\pi r}{360} = \frac{\pi r}{180} \tag{4}$$

An arc length of $n^\circ$ is

$$p_{n^\circ} = \frac{\pi r n}{180} \tag{5}$$

Formulas (2) to (5)—all of them are readily derived from (1)—are of great theoretical and practical value.

**Example 1.** A 2.4-metre strip of steel is used to make a hoop; 0.2 metre is taken up at the ends for riveting. What is the radius of the hoop?

The length of the circumference $p = 2.4 - 0.2 = 2.2$ metres. By formula (3)

$$r = \frac{p}{2\pi} \approx \frac{2.2}{6.3} \approx 0.35 \text{ m}$$

**Example 2.** The diameter of the driving wheel of a locomotive is 1.5 metres. How many rotations per minute does the wheel make when the train's speed is 30 km/hr?

The wheel covers per minute a distance of $30:60 = \frac{1}{2}$ (km), or 500 metres. In one rotation, it covers a distance equal to

the length of the circumference $p$; $p = \pi d \approx 3.14 \cdot 1.5 \approx 4.71$ metres. The desired number of rotations is $500 : 4.71 \approx 106$.

**Example 3.** A railway line has a track curvature of 800 metres radius. The track length on this curvature is 60 metres. How many degrees are there in the arc of the curvature?

By formula (5),

$$n = \frac{180p}{\pi r} \approx \frac{180 \cdot 60}{3.14 \cdot 800} \approx 4°20' \text{ (rounded)}$$

The *area of a circle* is equal to the product of half the circumference by the radius:

Fig. 122.

$$S = \frac{1}{2} pr \quad \text{or} \quad S = \pi r^2$$

The *area of a sector* of a circle ($S_{sect}$) is equal to the product of half the arc length ($p_{sect}$) by the radius ($r$):

$$S_{sect} = \frac{1}{2} p_{sect} r$$

The area of a sector with an arc of $n°$ is

$$S_{n°} = \frac{\pi r^2 n}{360}$$

The area of a segment of a circle is found as the difference between the area of the sector $AOBm$ (Fig. 122) and the triangle $AOB$.

### 152a. Huygens' Formula for Arc Length

In practical situations, one often has to find the length of an arc given in a drawing (or full-size) when it is not known what part of the circle the arc constitutes and what

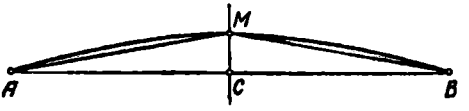


Fig. 122a.

the radius of the circle is. In such cases, use can be made of the following device.

Mark the midpoint $M$ (Fig. 122a) of a given arc $\overset{\frown}{AB}$ (this point lies on the perpendicular $CM$ drawn to the midpoint $C$

of chord $AB$). Then measure the chord $AB$ and the chord $AM$ which intercepts half the arc. The length $p$ of arc $\overset{\frown}{AB}$ is given (approximately) by the following formula of Huygens: *

$$p \approx 2l + \frac{1}{3}(2l - L)$$

where $l = AM$ and $L = AB$.

This formula admits of a relative error of about 0.5%, when $\overset{\frown}{AB}$ subtends 60°. The percent of error falls off sharply as the angular measure of the arc decreases. Thus, for an arc of 45° the relative error constitutes approximately 0.02%.

**Example.** Figure 122*a* depicts an arc $\overset{\frown}{AB}$ for which

$$l = AM = 34.0 \text{ mm}, \quad L = AB = 67.1 \text{ mm}$$

Huygens' formula yields

$$p = 2 \cdot 34.0 + \frac{1}{3}(2 \cdot 34.0 - 67.1) \approx 68.3 \text{ (mm)}$$

Here, all figures are correct since the arc $\overset{\frown}{AB}$ subtends about 45° (this can be gauged by eye) and, hence, the error of the formula comes to roughly 0.02%, which is less than 0.05 mm.

### 153. Measuring Angles in a Circle

An *inscribed angle* constitutes one-half the central angle of the intercepted arc. In Fig. 123, $\angle ACB = \frac{1}{2} \angle AOB$.

Fig. 123.

Fig. 124.

Fig. 125.

Therefore all inscribed angles intercepting the same arc are equal. In Fig. 124, $\angle ACB = \angle ADB = \angle AEB$. In other words, the chord $AB$ is seen from the same angle from all points of the intercepted arc. We say that arc $ACDEB$ subtends an angle of a definite magnitude. For example, a semicircle subtends an angle of 90° (Fig. 125).

* Christian Huygens (1629–1695), Dutch scientist noted for works in the fields of optics and mechanics.

Since the central angle contains as many angle degrees as arc degrees, it follows that an *inscribed angle* ($\angle ACB$ in Fig. 123) is measured by one half the intercepted arc $AB$.

An angle formed by two chords (say, $\angle AOB$ in Fig. 126) is measured by half the sum of the arcs, $\frac{1}{2}(\overset{\smile}{CD}+\overset{\smile}{AB})$, contained between its two sides (produced in both directions). An inscribed angle is a special case of the one under consideration (one of the arcs is equal to zero).

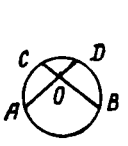


Fig. 126

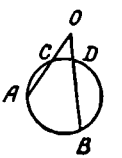


Fig. 127.

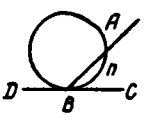


Fig. 128.

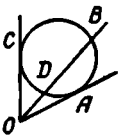


Fig. 129.

An *angle formed by two secant lines* ($\angle AOB$, Fig. 127) is measured by one-half the difference of the arcs lying between its two sides: $\frac{1}{2}(\overset{\smile}{AB}-\overset{\smile}{CD})$. An inscribed angle is a particular case of the angle between two secant lines ($\overset{\smile}{CD}=0$).

Regarding a tangent line as a degenerate secant line (Sec. 151), we find that the *angle formed by a tangent line and a chord* (say, $ABC$ in Fig. 128), is measured by one-half the intercepted arc $\left(\frac{1}{2}\overset{\smile}{AnB}\right)$; an *angle formed by a tangent line and a secant line* (say, $\angle BOA$ in Fig. 129) is measured by one-half the difference of the arcs between its sides: $\frac{1}{2}(\overset{\smile}{BA}-\overset{\smile}{DA})$; a *circumscribed angle* ($\angle COA$ in Fig. 129) is measured by one-half the difference of the arcs between its sides: $\frac{1}{2}(\overset{\smile}{CBA}-\overset{\smile}{CDA})$.

### 154. Power of a Point

The *power of a point* $O$ with reference to a given circle of radius $r$ is the quantity $d^2-r^2$, where $d$ is the distance $OC$ from the point to the centre of the circle. The power of a point external to the circle is positive, that of a point internal to the circle, negative. For points on the circumference, the power is zero.

The absolute value of the power of a point, $|d^2-r^2|$, is denoted by $p^2$ so that for an exterior point, $p^2=d^2-r^2$, and for an interior point, $p^2=r^2-d^2$. The quantities $p^2$ and $p$ (the latter is assumed to be positive) play an important role.

Let there be drawn through $O$ (Figs. 130 and 131) all kinds of secant lines ($AB$, $DE$, $FG$, etc.). The product of the length of a secant from $O$ to its point of intersection with the circle ($OA \cdot OB$ or $OD \cdot OE$ or $OF \cdot OG$, etc.) is a constant equal to $p^2$. Of particular importance is the case when the secant line passes through the centre $C$ (see examples below).

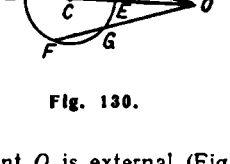


Fig. 130.

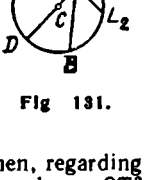


Fig 131.

If point $O$ is external (Fig. 130), then, regarding the tangent line as a degenerate secant line, we have $OT^2=p^2$, or the *absolute value of the power of a point is the square of the length of the tangent line*. Thus, the quantity $p$ is equal to the length of the tangent line $OT$.

If point $O$ is internal (Fig. 131), then, drawing through $O$ the chord $L_1L_2$ perpendicular to the diameter $DE$, we have $OL_1=OL_2$ so that $OL_1^2=p^2$, or the power of a point is equal to the square of the least semichord passing through the point. Thus, the quantity $p$ is equal to the length of the semichord $OL_1$.

**Example 1.** What is the range of sight from an aircraft flying over the sea at an altitude of 2 km? Take the diameter of the earth at 12,700 km.

Figure 132 gives a schematic vertical cross section of the earth with $O$ as the location of the aircraft, $OE=2$ km, $ED \approx 12{,}700$ km. The point farthest away on the earth's surface as seen from the airplane is $T$; $OT$ is the tangent line to the circle $ETD$; $OT=p$. On the other hand, $p^2=OE \cdot OD \approx 2 \cdot 12{,}700$ (we take $OD \approx 12{,}700$ km, dropping 2 km as a quantity definitely less than the limiting error of the approximate value of 12,700 km). We thus get

$$p=\sqrt{25{,}400} \approx 160 \text{ (km)}$$

**Example 2.** The span of a masonry vault is 6 metres; it has an altitude of 0.4 metre. Determine the radius of the arc of the vault.

Fig. 132.

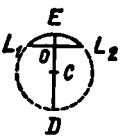


Fig. 133.

See Fig. 133 (schematic): $L_1L_2=6$ m, $EO=0.4$ m. The power of the point $O$ is $p^2=OL_1^2=\left(\frac{L_1L_2}{2}\right)^2=9$. On the other hand, $p^2=EO\cdot OD$; since $EO$ is small compared with $OD$, we can take $OD=2r$ and we get $9\approx 0.4\cdot 2r$, whence

$$r=\frac{9}{0.8}\approx 11\,\frac{1}{4}\,(\text{m})$$

### 155. Radical Axis. Radical Centre

The locus of points $M$ (Figs. 134, 135, 136, 137, 138) having equal powers with reference to two given circles $O_1$, $O_2$ ($MK_1=MK_2$) is a straight line $AB$ perpendicular to the line of the centres.

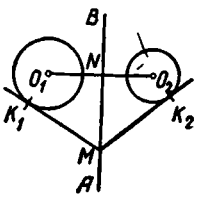


Fig. 134.

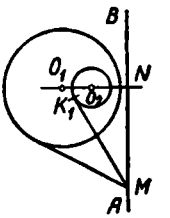


Fig. 135.

This straight line is called the *radical axis* of the circles $O_1$ and $O_2$. The distances $d_1$, $d_2$ of the radical axis from the centres $O_1$, $O_2$ of the given circles may be computed by the

formulas

$$d_1 = O_1N = \frac{d}{2} + \frac{r_1^2 - r_2^2}{2d}$$

$$d_2 = NO_2 = \frac{d}{2} + \frac{r_2^2 - r_1^2}{2d}$$

where $d$ is the distance $O_1O_2$ between the centres of the circles, and $r_1$ and $r_2$ are the radii of the circles. It is much

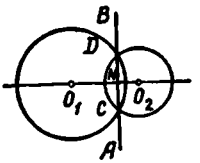


Fig. 136.

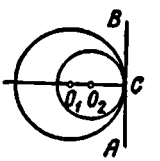


Fig. 137

easier to find the radical axis by means of a construction. If the circles $O_1$ and $O_2$ intersect at points $C$ and $D$, then each of these points has power zero with respect to both circles and, hence, the radical axis passes through $C$ and $D$ (Fig. 136). If the circles touch at point $C$ (Fig. 137, 138), then the common tangent serves as their radical axis. For

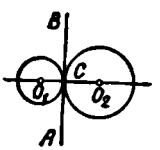


Fig. 138.

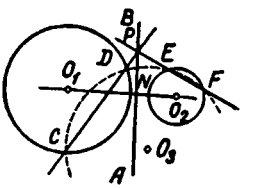


Fig. 139.

nonintersecting circles, the radical axis may be found thus. Construct (Fig. 139) an auxiliary circle $O_3$ to intersect circle $O_1$ in points $C$ and $D$ and circle $O_2$ in points $E$ and $F$. The straight lines $CD$, $EF$ are the radical axes of the two pairs of circles $O_1$, $O_3$ and $O_2$, $O_3$. For this reason, their point $P$ of intersection has the same power with respect to $O_1$, $O_3$ as it has with respect to $O_2$, $O_3$. Hence, it has the same powers

with respect to $O_1$ and $O_2$, that is to say, it lies on the radical axis of the two given circles. Finding another point in the same manner or dropping a perpendicular $PN$ from $P$ to $O_1O_2$, we find the required radical axis.

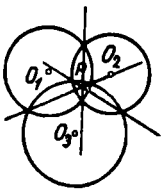


Fig. 140.

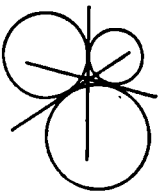

Fig. 141.

The foregoing reasoning shows that the three radical axes of any three circles $O_1$, $O_2$, $O_3$ (taken in pairs) intersect in one point. This point is termed the *radical centre* of the circles $O_1$, $O_2$, $O_3$. In particular, the three common chords of the three pairwise intersecting circles in Fig. 140 intersect in a single point. The three common tangent lines of three pairwise tangent circles (Fig. 141) also intersect in a single point.

### 156. Inscribed and Circumscribed Polygons

A *polygon inscribed in a circle* is one in which all the vertices lie on the circumference (Fig. 142); a polygon circumscribed about a circle is one in which the sides are tangent to the circle (Fig. 143).

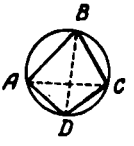


Fig. 142.

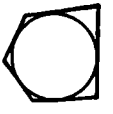

Fig. 143.

A *circle circumscribed about a polygon* is one that passes through the vertices (Fig. 142); a *circle inscribed in a polygon* is one which touches the sides of the polygon (Fig. 143). If we take an arbitrary polygon, a circle can neither be inscribed

in it nor circumscribed about it. In the case of a triangle, it is always possible to inscribe a circle in it and circumscribe a circle about it (see Sec. 137 Problems 20, 21 and Sec. 146).

The radius $r$ of an inscribed circle is given in terms of the sides $a$, $b$, $c$ of a triangle by the formula

$$r=\sqrt{\frac{(p-a)\,(p-b)\,(p-c)}{p}}$$

where

$$p=\frac{a+b+c}{2}$$

The *radius R of a circumscribed circle* is given by the formula

$$R=\frac{abc}{4\sqrt{p\,(p-a)\,(p-b)\,(p-c)}}$$

A circle can be inscribed in a quadrilateral only when the sums of the opposite sides are equal; of all parallelograms, only the rhombus (or square, as a special case) admits inscribing a circle. Its centre lies at the intersection of the diagonals.

A circle can be circumscribed about a quadrilateral only when the sum of the opposite angles is equal to 180° (if this is true for one pair of opposite angles, then the sum of the other pair of angles will definitely come to 180°). Of all parallelograms only the rectangle (square, as a special case) admits circumscribing a circle; its centre lies at the intersection of the diagonals.

A circle can be circumscribed about a trapezoid only when it is an isosceles trapezoid.

In a convex quadrilateral inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides (Ptolemy's theorem). In Fig. 142

$$AC\cdot BD=AB\cdot DC+AD\cdot BC$$

### 157. Regular Polygons

A *regular polygon* is a polygon with equal sides and angles. Figure 144 shows a regular hexagon and Fig. 145 a regular octagon. A regular quadrilateral is a square, and a regular triangle is an equilateral triangle. Each angle of a regular $n$-gon is equal to $\frac{180^\circ\,(n-2)}{n}$.

Referring to Fig. 144 we see that a regular polygon has a point $O$ inside it that is equidistant from all the vertices ($OA = OB = OC$, etc.). This is the centre of the regular polygon. The centre is also equidistant from all the sides of a regular polygon ($OP = OQ = OR$, etc.).

The lines $OP$, $OQ$, etc. are called *apothems* (or short radii), and the lines $OA$, $OB$, etc., are called the *radii* of a regular polygon.

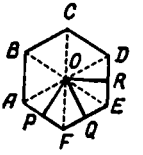


Fig. 144.

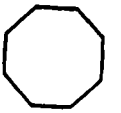

Fig. 145.

The area of a regular polygon is equal to the product of the semiperimeter by an apothem:

$$S = ph$$

where

$$p = \frac{1}{2}(AB + BC + CD + \ldots), \quad h = OP$$

A circle may be inscribed in a regular polygon or circumscribed about it. The centres of the inscribed and circumscribed circles lie at the centre of the regular polygon. The radius of a circumcircle is the radius of the regular polygon; the radius of the inscribed circle is the apothem. (For the construction of inscribed and circumscribed circles of polygons see Sec. 137, Problems 30-38). A side $b_n$ of a regular circumscribed polygon can be expressed in terms of the side $a_n$ of a regular inscribed polygon with the same number of sides by the formula

$$b_n = Ra_n : \sqrt{R^2 - \frac{1}{4}a_n}$$

where $R$ is the radius of the circle.

The side $a_{2n}$ of a regular inscribed polygon with double the number of sides is expressed in terms of $a_n$ by the formula

$$a_{2n} = \sqrt{2R^2 - 2R\sqrt{R^2 - \frac{1}{4}a_n^2}}$$

The following formulas express relationships between the sides of certain regular inscribed polygons and the radius of the circle:

$$a_3 = R\sqrt{3} \approx 1.7321R,$$

$$a_4 = R\sqrt{2} \approx 1.4142R,$$

$$a_5 = R\sqrt{\frac{5-\sqrt{5}}{2}} \approx 1.1755R,$$

$$a_6 = R,$$

$$a_8 = R\sqrt{2-\sqrt{2}} \approx 0.7654R,$$

$$a_{10} = R\frac{\sqrt{5}-1}{2} \approx 0.6180R,$$

$$a_{12} = R\sqrt{2-\sqrt{3}} \approx 0.5176R,$$

$$a_{15} = \frac{1}{4}R\left[\sqrt{10+2\sqrt{5}} - \sqrt{3}\,(\sqrt{5}-1)\right] \approx 0.4158R$$

The expressions for $a_3$, $a_4$ and $a_6$ are frequently used in practical computations; it is best to remember them. The most convenient method for computing the sides of the other polygons is by means of the formulas of trigonometry (see Sec. 190) using tables. For most polygons, the relations $a_n : R$ cannot be expressed as algebraic formulas even if we use nested radical signs (radicals within radicals).

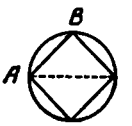

Fig. 146.

**Example.** Is it possible to obtain a square beam (36 cm on a side) from a log 40 cm in diameter?

We can take the cross section of the log as a circle of radius

$$R = \frac{40}{2} = 20 \text{ (cm)}$$

The largest square in this circle is the square that is inscribed in it. Its side $AB$ (Fig. 146) is equal to $20\sqrt{2} \approx 20 \cdot 1.41 \approx 28$ (cm). Hence, we cannot obtain a $36 \times 36$ cm beam from this log.

### 158. Areas of Plane Figures

In this section we give the most important formulas for finding the areas $S$ of plane figures (some of them have already been given in appropriate sections).

*Square* (Fig. 103, page 274): $a$ is a side, $d$, a diagonal:

$$S = a^2 = \frac{d^2}{2}$$

*Rectangle* (see Fig. 101, page 274): $a$ and $b$ are sides:

$$S = ab$$

*Rhombus* (see Fig. 102, page 274): $a$ is a side, $d_1$ and $d_2$ are diagonals, and $\alpha$ is one of the angles (acute or obtuse):

$$S = \frac{d_1 d_2}{2} = a^2 \sin \alpha$$

*Parallelogram* (see Fig. 100 page 274): $a$ and $b$ are sides, $\alpha$ is one of the angles (acute or obtuse) and $h$ is the altitude:

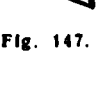

Fig. 147.

$$S = ah = ab \sin \alpha$$

*Trapezoid* (see Figs. 104, 106, page 275): $a$ and $b$ are bases, $h$, altitude, and $c$, the median:

$$S = \frac{a+b}{2} h = ch$$

*Any quadrilateral:* $d_1$, $d_2$, diagonals, $\alpha$, the angle between them (Fig. 147):

$$S = \frac{1}{2} d_1 d_2 \sin \alpha$$

A *circumscribable quadrilateral* (Sec. 137, Problem 22) $a$, $b$, $c$, $d$ are the sides:

$$p = \frac{a+b+c+d}{2}$$

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$$

*Right triangle* (see Fig. 75, page 266): $a$ and $b$ are the legs:

$$S = \frac{1}{2} ab$$

An *isosceles triangle* (see Fig. 77, page 267): $a$, the base, $b$, a side:

$$S = \frac{1}{2} a \sqrt{b^2 - \frac{a^2}{4}}$$

An *equilateral triangle* (see Fig. 78, page 267): $a$, a side:

$$S = \frac{1}{4} a^2 \sqrt{3}$$

An *arbitrary triangle* with $a$, $b$, $c$ the sides, $a$ the base, $h$ the altitude, and $A$, $B$, $C$ the angles opposite the sides $a$, $b$, $c$, respectively; $p=\frac{a+b+c}{2}$ (Fig. 148):

$$S=\frac{1}{2}ah=\frac{1}{2}ab\sin C=\frac{a^2\sin B\sin C}{2\sin A}=\frac{h^2\cdot\sin A}{2\sin B\sin C}$$
$$=\sqrt{p(p-a)(p-b)(p-c)}$$

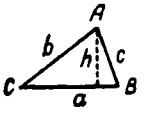


Fig. 148.

Fig. 149.

A *polygon* whose area is sought is partitioned into triangles in any manner (say by diagonals). It is convenient to partition a polygon, circumscribed about a circle, by straight lines from the centre of the circle to the vertices of the polygon (Fig. 149). We then get

$$S=rp$$

where $r$ is the radius of the circle and $p$ is the semiperimeter.

In particular, this formula is valid for any regular polygon.

A *regular hexagon* in which $a$ is a side:

$$S=\frac{3}{2}\sqrt{3}a^2$$

A *circle* in which $d$ is the diameter, $r$ the radius, $C$ the length of the circumference:

$$S=\frac{1}{2}Cr=\pi r^2\ (\approx 3.142r^2)=\pi\frac{d^2}{4}\ (\approx 0.785d^2)$$

A *sector* of radius $r$; $n$ is the degree measure of the central angle and $p_{n^\circ}$ is the arc length (Fig. 150):

$$S=\frac{1}{2}rp_{n^\circ}=\frac{\pi r^2 n}{360}$$

An *annulus* (Fig. 151) where $R$ and $r$ are the outer and inner radii, respectively; $D$ and $d$, the outer and inner diameters, respectively; $\bar{r}$ the mean radius; and $k$ the width of

the annulus:

$$S=\pi(R^2-r^2)=\frac{\pi}{4}(D^2-d^2)=2\pi\bar{r}k$$

*Segment.* The area of a segment (Fig. 152) is found as the difference between the area of the sector $OAmB$ and the triangle $AOB$.

Fig 150.

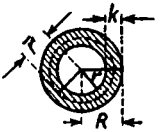


Fig. 151.

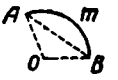


Fig 152.

### 158a. Approximate Formula for the Area of a Segment

In practical situations, one often has to find the area of a segment given natural-size or in a drawing when it is not known what part of the circle the arc of the segment constitutes and what its radius is. In such cases the following approximate formula may be used:

$$S\approx\frac{2}{3}ah$$

where $a=AB$ (Fig. 152*a*) is the base of the segment, $h=CM$ is its altitude. In other words, it is taken that the area of a segment is equal to $\frac{2}{3}$ the rectangle $ADEB$. Actually, the

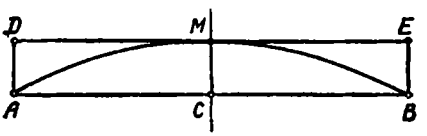


Fig. 152a.

area of the segment is somewhat greater. For $\overset{\smile}{AB}=60°$ the relative error of the formula comes to 1.5%, for $\overset{\smile}{AB}=45°$, it is half as much, for $\overset{\smile}{AB}=30°$ it is 0.3% and continues to fall off even more rapidly.

**Example.** Find the area of the segment $AMB$ (Fig. 152*a*) whose base $a = 60.0$ mm and altitude $h = 8.04$ mm.

**Solution.** $S \approx \frac{2}{3} \cdot 60.0 \cdot 8.04 \approx 321$ (mm²). However, the third digit is definitely incorrect since the arc $\overset{\smile}{AB}$ contains 60° (this is visibly so) and, hence, the error of the formula comes to 1.5%, which is roughly 5 square millimetres. Making the appropriate correction, we find $S \approx 326$ mm²; all digits are true.

## B. SOLID GEOMETRY

### 159. General

*Solid geometry* studies the geometric properties of objects and figures in space. When solving problems in solid geometry, an important technique is the consideration of plane lines and figures (both those in the solids under study and also lines and figures constructed as auxiliary elements). It is therefore very important to learn to recognize and isolate diverse plane figures in spatial images.

### 160. Basic Concepts

As in plane geometry the most elementary line is the straight line, so in solid geometry the most elementary surface is the plane surface—the *plane*. *Planes* and *straight lines* are the basic elements of solid geometry.

One and only one plane can be drawn through any three points of space not all on one straight line. Through three points of one straight line, it is possible to pass an infinitude of planes, called a *pencil* of planes; the line through which all the planes pass is called the *axis* of the pencil.

One and only one plane can be drawn through any straight line and a point external to the line. It is not always possible to draw a plane through two straight lines. Two straight lines through which it is impossible to draw a plane are termed *skew lines*.

**Example.** A horizontal line drawn on one wall of a room and a vertical line drawn on the opposite wall are skew lines.

Skew lines are nonintersecting lines no matter how far produced, but they are not parallel.

*Parallel lines* are nonintersecting lines such that a plane can be passed through them (cf. Sec. 148).

The difference between parallel lines and skew lines is that two parallel straight lines have the same direction, while skew lines have different directions.

All the points of one of two parallel lines are equidistant from the corresponding points of the other (the perpendicular distance is taken), whereas the points of one of two skew lines are at different distances from the corresponding points of the other skew line.

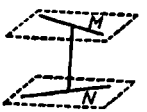

Fig. 153.

One and only one plane can be drawn through two intersecting straight lines.

The *distance between two skew lines* is given by the length of the line segment $MN$ (Fig. 153) joining the nearest points $M$ and $N$ of the skew lines. The straight line $MN$ is perpendicular to both skew lines.

The distance between parallel straight lines is defined as in plane geometry. The distance between intersecting straight lines is taken to be zero.

Two planes may intersect (along a straight line) or may not intersect. Nonintersecting planes are termed *parallel planes.*

A straight line and a plane also either intersect (in a point) or do not intersect, in which case we say that *the straight line is parallel to the plane* (or that the plane is parallel to the straight line).

### 161. Angles

The *angle between two intersecting straight lines* is measured in the same way as in plane geometry (because a plane can be drawn through two such straight lines). The *angle between parallel lines* is taken to be zero (or 180°; see Sec. 148). The angle between skew lines $AB$ and $CD$ (Fig. 154) * is defined thus: through any point $O$ draw rays $OM \parallel AB$ and $ON \parallel CD$. The angle between $AB$ and $CD$ is taken to be equal to the angle $NOM$. In other words, the straight lines $AB$ and $CD$ are translated parallel to themselves to new positions until they intersect. For instance, we can take $O$ on one of the lines $AB$, or $CD$, which will then remain fixed.

* On the straight line $AB$ (and on $CD$) we can establish a direction at pleasure: from $A$ to $B$ or from $B$ to $A$ (from $C$ to $D$ or from $D$ to $C$). In the former case, the line is denoted $AB$, in the latter, $BA$.

The straight line $AB$ that intersects plane $P$ at point $O$ forms, generally speaking, distinct angles with distinct lines $OC$, $OD$, $OE$ drawn in the plane $P$ through $O$ (angles $AOC$, $AOD$, $AOE$, Fig. 155). If $AB$ is perpendicular to two such straight lines (say $OE$, $OD$), then it is perpendicular to all

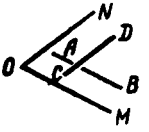

Fig. 154.

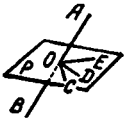

Fig. 155.

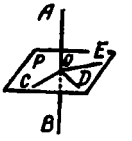

Fig. 156.

the other lines passing through $O$ (say, $OC$). In this case we say that the straight line $OA$ (Fig. 156) is *perpendicular* to the plane $P$, and the plane $P$ is perpendicular to the straight line $OA$.

*The orthogonal projection* (or simply, projection) *of a point A on a plane P* is the foot $C$ of the perpendicular drawn

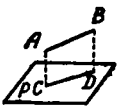

Fig. 157.

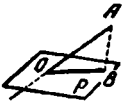

Fig. 158.

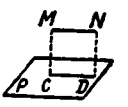

Fig. 159.

from the point $A$ to the plane $P$. The projection of a line-segment $AB$ on a plane $P$ is the line segment $CD$ whose endpoints are projections of the extremities of $AB$ (Fig. 157). Operations involving projections constitute one of the basic techniques of geometric investigation (see Sec. 162). The angle between a straight line and a plane is determined by means of a projection operation.

In Fig. 158, *the angle between the straight line OA and plane P* is the angle formed by $OA$ and its projection $OB$ in the $P$ plane. If the straight line $MN$ is parallel to the $P$ plane (Fig. 159), then it is parallel to its projection and the (acute) angle between $MN$ and the $P$ plane is taken to be zero.

The figure (see Fig. 160) formed by two half-planes $P$ and $Q$ emanating from one straight line $CD$ is called a *dihed-*

*ral* angle. The line $CD$ is the edge of the dihedral angle, the planes $P$ and $Q$ are called the faces of the angle.

Plane $R$, which is perpendicular to the edge of the dihedral angle, forms an angle $AOB$ at the intersection with the faces $P$ and $Q$. This is called the *plane angle* of the dihedral angle.

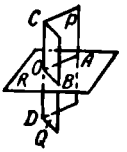


Fig. 160.

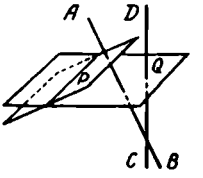


Fig. 161.

A dihedral angle is measured by its plane angle. Instead of saying that the "measure of a dihedral angle is 30°" we say that the "dihedral angle is equal to 30°", and the like.

One often also speaks of "an angle between two planes" (cf. with plane geometry where one speaks of "an angle between two lines"). This angle is then one of the four angles formed by the planes (ordinarily, the acute angle).*

*The angle* (acute) *between two parallel planes* is said to be zero. (Actually there is no angle at all.)

Two planes that form a right angle are termed perpendicular.

In Fig. 161, the angles formed by the two straight lines $AB$ and $CD$, which are respectively perpendicular to the planes $P$ and $Q$, are equal to the angles between $P$ and $Q$ (acute angles being equal to acute angles, and obtuse angles to obtuse angles). For this reason, the measure of an angle between two planes, $P$ and $Q$, may also be defined as the value of the angle formed by the straight lines $AB$ and $CD$.

### 162. Projections

Any line (and not only a straight line), whether it lies in a plane or does not, can be projected onto a plane. In Fig. 162, let $ABCDE$ be a line (curved or polygonal). Let

* Vertical and adjacent dihedral angles are defined in the same way as vertical and adjacent angles between straight lines. Vertical dihedral angles are equal.

us move a point along this line; as the point takes up positions $A$, $B$, $C$, $D$, etc., its projection (Sec. 161) will take up the positions $a$, $b$, $c$, $d$, etc. The line *abcde* described by the projection of the point moving along $ABCDE$ is called the *projection* of the line $ABCDE$. Though the shape of the projection depends upon the shape of the line being projected, it does not determine the shape of the line being projected. However, if we know the projections of a line $ABCDE$ on two planes, then the shape of the line $ABCDE$ itself is defined (there are some special line arrangements that constitute exceptions). This fact underlies the method of *descriptive geometry* in which a geometric figure is studied on the basis of its projections on two mutually perpendicular planes.

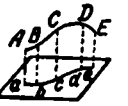

Fig. 162.

The shape of a line projected onto a plane changes. Thus, for instance, if we project (see Fig. 163) a circle on plane $P$ from plane $Q$, which is not parallel to $P$, the projection will be an oval curve called an *ellipse*.

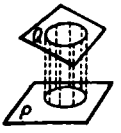

Fig. 163.

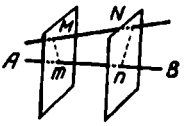

Fig 164.

If a closed curve lying in the $Q$ plane is projected on the $P$ plane, then the area $S_1$ bounded by the projection is connected with the area $S$ bounded by the figure being projected via the relation

$$S_1 = S \cos \alpha$$

where $\alpha$ is the angle between the planes $P$ and $Q$.

A similar formula relates the length $a$ of $AB$ (see Fig. 157 on page 297) to the length $a_1$ of its projection $CD$ on the plane $P$:

$$a_1 = a \cos \alpha$$

where $\alpha$ is the angle between the straight line $AB$ and the plane $P$.

Points and lines are often projected onto a *straight line (the axis of projections)*.

Suppose we have a straight line $AB$ and a point $M$ (Fig. 164). Draw through $M$ a plane perpendicular to $AB$. It intersects $AB$ in a point $m$; the point $m$ is called the *projection of point M on the line AB*.

Projecting the extremities $M$ and $N$ of line $MN$ on the line $AB$, we obtain the points $m$ and $n$; the segment they bound is called the projection of the line-segment $MN$ on the straight line $AB$.* The length $a$ of $MN$ is connected with the length $a_1$ of its projection $mn$ by the formula

$$a_1 = a \cos \alpha$$

where $\alpha$ is the angle between the lines $MN$ and $AB$. The projections of line segments on a straight line may be considered algebraic quantities in the same way as in making projections in a single plane (see Sec. 147). Then a theorem similar to the theorem of plane geometry holds true: the sum of the projections of the segments of a polygonal line is equal to the projection of the closing segment.

### 163. Polyhedral Angles

If through a given point $O$ (Fig. 165) are passed planes $AOB$, $BOC$, $COD$, etc. that intersect one another successively along the straight lines $OB$, $OC$, $OD$, etc. (the last plane $AOE$ intersects the first along the line $OA$), then the resulting

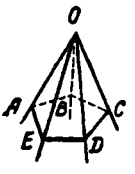

Fig. 165.

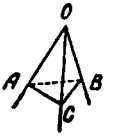

Fig. 166.

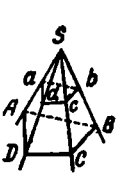

Fig. 167

figure is called a *polyhedral angle*. The point $O$ is the *vertex* of the polyhedral angle.

The planes which form a polyhedral angle are called *faces*. The straight lines along which the faces intersect in succes-

* Note that $Mm$ and $Nn$ are perpendicular to $AB$, but in the general case (cf. Fig. 153 on page 296) they are not parallel to one another: they are skew lines if the straight lines $AB$ and $MN$ are skew lines.

sion are termed the *edges* of the polyhedral angle. The angles *AOB*, *BOC* and so on are called *face angles*.

The smallest number of faces of a polyhedral angle is three (the *trihedral angle* in Fig. 166). Each face angle of a trihedral angle is less than the sum and greater than the difference of the other two face angles.

The section of a polyhedral angle by a plane (not passing through the vertex) is a polygon (*ABCDE* in Fig. 165).* If it is convex, the polyhedral angle is termed *convex*. In a convex polyhedral angle the sum of the face angles does not exceed 360°.

Parallel planes intercept, on the edges of a polyhedral angle (Fig. 167), proportional segments ($SA:Sa=SB:Sb$, etc.) and form similar polygons (*ABCD* and *abcd*).

### 164. Polyhedrons: Prism, Parallelepiped, Pyramid

A *polyhedron* is a solid bounded by plane surfaces called faces (polygons). The faces intersect in straight lines called *edges*. Their vertices are *the vertices of the polyhedron*. Lines connecting two vertices not lying in one face are called *diagonals* of the polyhedron. A *convex polyhedron* is one in which all diagonals lie within it.

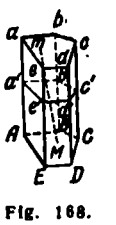

Fig. 168.

Fig. 169.

A *prism* (Fig. 168) is a polyhedron in which two faces *ABCDE* and *abcde* (the bases of the prism) are congruent polygons with corresponding sides parallel, and all other faces (*AabB*, *BbcC*, etc.) are parallelograms whose planes are parallel to one straight line (*Aa*, or *Bb*, or *Cc*, etc.) The parallelograms *ABba*, *BCcb*, etc. are *lateral faces*. The edges *Aa*, *Bb*, etc. are called *lateral edges*. The *altitude* of a prism (*Mm* in Fig. 168) is the perpendicular distance between the bases. A prism is triangular, quadrangular, pentagonal, hexagonal, octagonal, etc. according as the base is a triangle, quadrangle (quadrilateral), pentagon, hexagon, octagon, etc.

* Elementary geometry considers only polyhedral angles such that the contour *ABCDE* has no self-intersections. A simple polyhedral angle isolates a portion of space which is also called a polyhedral angle. On the measurement of polyhedral angles see Sec. 172.

If the lateral edges of the prism are perpendicular to the bases, we have a *right prism*, otherwise it is an *oblique prism*. If the bases in a right prism are regular polygons, the prism is a *regular prism*. Figure 168 is an oblique pentagonal prism, Fig. 169, a regular hexagonal prism.

A *right section of a prism*, $a'b'c'd'e'$, is a plane section perpendicular to a lateral edge (Fig. 168).

The lateral area of a prism is equal to the length ($l$) of a lateral edge times the perimeter ($p'$) of a right section:

$$S_{lat} = p'l$$

For a right prism, the bases are right sections and the altitude $h$ is equal to a lateral edge, so that

$$S_{lat} = ph$$

The volume ($V$) of a prism is equal to the product of the area ($S'$) of a right section by the length ($l$) of a lateral edge:

$$V = S'l$$

or the product of the area ($S$) of the base by the altitude:

$$V = Sh$$

A *parallelepiped* is a prism whose bases are parallelograms (Fig. 170). Thus, a parallelepiped has six faces and they are

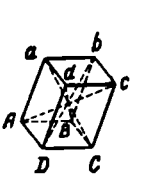

Fig. 170.

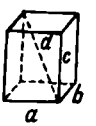

Fig. 171

all parallelograms. Opposite faces are equal and parallel. A parallelepiped has four diagonals which all intersect in one point and are bisected by this point. Any face may be taken as the base. The volume is equal to the area of the base times the altitude:

$$V = Sh$$

A *right parallelepiped* is a parallelepiped whose four lateral faces are rectangles.

A right parallelepiped, all six faces of which are rectangles, is called a rectangular parallelepiped (Fig. 171). The volume ($V$) of a right parallelepiped is equal to the area ($S$) of a base times the altitude ($h$):

$$V = Sh$$

For a rectangular parallelepiped we also have the formula

$$V = abc$$

where $a$, $b$, $c$ are edges.

The diagonal ($d$) of a rectangular parallelepiped is related to its edges by

$$d^2 = a^2 + b^2 + c^2$$

A *cube* is a rectangular parallelepiped with all the faces squares. All the edges of a cube are equal; the volume ($V$) of a cube is given by the formula

$$V = a^3$$

where $a$ is an edge of the cube.

A *pyramid* is a polyhedron with one face (the *base*) an arbitrary polygon ($ABCDE$ in Fig. 172), and the other faces

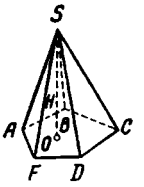

Fig. 172.

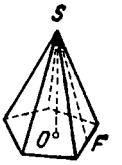

Fig. 173.

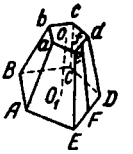

Fig. 174

(*lateral faces*) triangles with a common vertex $S$, called the *vertex* of the pyramid. The perpendicular $SO$ from the vertex to the base is the *altitude* of the pyramid. A pyramid is triangular, quadrangular, pentagonal, etc. according as the base is a triangle, quadrangle, pentagon, etc. A triangular pyramid is a tetrahedron, a quadrangular pyramid is a pentahedron, etc.

A pyramid is called *regular* if its base is a regular polygon (Fig. 173) and the altitude falls in the centre of the base. In a regular pyramid, all lateral edges are equal because all lateral faces are congruent isosceles triangles. The alti-

tude ($SF$) of a lateral face of a regular pyramid is called the ***slant height.***

The lateral area of a regular pyramid is equal to the semiperimeter of the base $\left(\frac{1}{2}p\right)$ times the slant height ($a$):

$$S_{lat}=\frac{1}{2}pa$$

The volume of any pyramid is equal to one third the product of the area ($S$) of the base by the altitude ($h$):

$$V=\frac{1}{3}Sh$$

A section *abcde* drawn parallel to the base *ABCDE* (Fig. 174) produces a solid bounded by this section, the base, and the lateral surface between them; it is called the *frustum of a pyramid* (the term *truncated pyramid* is also used, though many authors restrict it to nonparallel bases). The parallel faces of the frustum (*ABCDE* and *abcde*) are called the bases; the distance between them ($OO_1$) is the altitude. A frustum is regular if the original pyramid was regular. All lateral faces of a regular frustum are congruent isosceles trapezoids. The altitude $Ff$ of a lateral face is called the *slant height* of the regular truncated pyramid.

The lateral area of a regular truncated pyramid is equal to the half-sum of the perimeters of the bases times the slant height:

$$S_{lat}=\frac{1}{2}(p_1+p_2)a$$

where $p_1$, $p_2$ are the perimeters of the bases and $a$ is the slant height.

The volume $V$ of any truncated pyramid is equal to one third the product of the altitude by the sum of the areas of the upper base, the lower base, and the mean proportional between them:

$$V=\frac{1}{3}h\left(S_1+\sqrt{S_1S_2}+S_2\right)$$

where $S_1$ is the area of *ABCDE*, $S_2$ is the area of *abcde*, and $h$ is the altitude $OO_1$.

As an illustration, the volume $V$ of a regular quadrangular truncated pyramid is given by the formula

$$V=\frac{1}{3}h(a^2+ab+b^2)$$

where $a$ and $b$ are the sides of the squares in the bases.

## 165. Cylinders

A *cylindrical surface* is a surface generated by a straight line ($AB$ in Fig. 175) moving in the same direction and intersecting a given line $MN$. The line $MN$ is called the *directrix* (plural, directrices). The straight line $AB$ is termed the *generator*, or *generatrix* (plural, generatrices), of the cylindrical surface. The generator in any one fixed position is called an *element*. A solid bounded by a cylindrical surface (with a closed-curve directrix) and two parallel planes is called a *cylinder* (Fig. 176). The portions of the parallel planes bounding the cylinder ($ABCDE$ and $abcde$) are termed the *bases* of the cylinder. The distance between the bases is the *altitude* of the cylinder ($MN$ in Fig. 176).

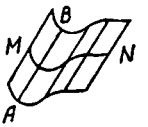

Fig. 175.

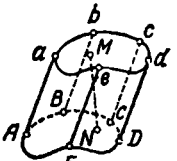

Fig. 176.

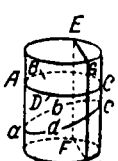

Fig. 177.

A prism is a special type of cylinder in which the elements of the generator are parallel to the lateral edges and the directrix is a polygon forming the bases. On the other hand, an arbitrary cylinder may be regarded as a degenerate (smoothed) prism with a very large number of very narrow faces. Practically speaking, a cylinder is indistinguishable from such a prism. All the properties of the prism are preserved in the cylinder (see below).

A cylinder is called a *right cylinder* if the elements (rulings) of the generator are perpendicular to the base, otherwise it is an oblique cylinder. A *circular cylinder* is one in which the base is a circle. A right circular cylinder (see Fig. 177) may be regarded as a degenerate regular prism. Right circular cylinders are commonplace in everyday life, as witness pipes, glasses, cans, etc. A right circular cylinder can be generated by revolving a rectangle about one of its sides, whence the alternative (synonymous) term *cylinder of revolution*.

Sections of the lateral surface of a circular cylinder (we assume the lateral surface to be produced beyond the bases of the cylinder) cut parallel to the bases ($ABCD$ in Fig. 177) are circles of the same radius. The sections parallel to the generator form pairs of parallel straight lines ($EF$ and $HG$).

Sections which are not parallel either to the bases or to the generator (*abcd*) are ellipses (see Sec. 162).

The lateral area of a cylinder is equal to the product of the generator by the length of the line bounding the section perpendicular to the generator. In a right cylinder, this section is the base, and the generator is the altitude. Therefore, the lateral area of a right circular cylinder is equal to the product of the circumference of a base by the altitude:

$$S_{lat} = 2\pi rh$$

The volume of any cylinder is equal to the product of the area of a base by the altitude:

$$V = Sh$$

For a right circular cylinder,

$$V = \pi r^2 h \quad (r \text{ the radius of a base})$$

### 166. The Cone

A *conical surface* is a surface generated by the motion of a straight line (*AB* in Fig. 178) which always passes through a fixed point (*S*) and intersects a given line (*MN*). In elementary geometry we consider only conical surfaces without self-intersections.

Fig. 178.

The line *MN* is called the *directrix*; the straight lines corresponding to various positions of *AB* are the elements of the generator of the conical surface. Point *S* is the *vertex* (or apex) of the cone. A conical surface has two sheets (nappes): one described by ray *SA*, the other, by its extension *SB*.

By conical surface we frequently mean one of its sheets.

A *cone* is a solid bounded by one sheet of a conical surface (with closed directrix) and a plane (*ABCDEFGHJ* in Fig. 179) intersecting it and not passing through the vertex *S*. The portion of this plane lying inside the conical surface is called the base of the cone. The perpendicular *SO* from the vertex to the base is the altitude of the cone.

A pyramid is a special case of a cone in which the directrix is a polygon; an arbitrary cone is a degenerate pyramid.

A cone is *circular* (Fig. 180) if its base is a circle.

The straight line *SO* connecting the vertex of the cone and the centre of the base is the *axis of the cone*. If the altitude of a circular cone falls in the centre of the base, then the cone is called a *right circular cone*, sometimes simply a circular cone (Fig. 181). A right circular cone can be generated by revolving a right triangle about one of its legs (hence the alternative name of *cone of revolution*).

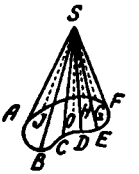


Fig. 179.

Fig. 180.

Fig 181.

A section of a circular cone by a plane parallel to the base is a circle (Fig. 180). See Sec. 167 on sections of a cone by planes not parallel to the base.

The lateral area of a right circular cone is equal to the product of one half the circumference of the base ($C$) by the generator ($l$):

$$S_{lat} = \frac{1}{2} Cl = \pi rl \quad (r \text{ the radius of the base})$$

The volume of any cone is equal to one third the product of the area ($S$) of the base by the altitude ($h$):

$$V = \frac{1}{3} Sh$$

For a right circular cone,

$$V = \frac{1}{3} Sh = \frac{1}{3} \pi r^2 h$$

### 167. Conic Sections

*Conics* (or *conic sections*) are lines of intersection of various planes with the lateral surface of a *circular* (but not necessarily right circular) cone. A conical surface is imagined as extending in both directions from the vertex without limit.

If a cutting plane intersects only one nappe of a circular cone and is not parallel to any element of the generator

(Fig. 182), then the conic section is an *ellipse* (see Sec. 162). In exceptional cases the ellipse becomes a circle (for instance, in a right circular cone all sections parallel to the base are circles).

If the cutting plane intersects only one nappe of a right circular cone and is parallel to one of the elements of the generator (Fig. 183), then the section yields an unbounded (in one direction) line called a *parabola*.

Fig. 182.

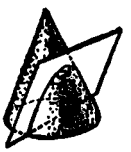

Fig. 183.

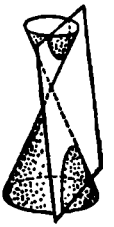

Fig. 184

If the cutting plane intersects both nappes of the surface of a circular cone (Fig. 184), then the section yields a line consisting of two branches receding without limit (*hyperbola*). In particular, a hyperbola is obtained when the cutting plane is parallel to the axis of the cone.

Conic sections are of both theoretical and practical interest. In the practical aspect we have elliptic gear wheels, parabolic searchlights; planets and certain comets move in elliptical orbits; some comets follow parabolic and hyperbolic paths.

The fundamentals of conic sections are given in all manuals devoted to analytic geometry.

### 168. The Sphere

A *spherical surface* (or, simply, *sphere*) is a locus of points of space equidistant from one point called the centre of the sphere (point $O$ in Fig. 185). The *radius* $OE$ and the *diameter* $EG$ of a spherical surface are defined in the same way as for a circle (Sec. 151).

*A solid bounded by a spherical surface is called a sphere.*

A sphere can be generated by revolving a semicircle (or a circle) about its diameter.

All plane sections of a sphere are circles (*ABCD* in Fig. 185). The radius of the circle increases as the cutting plane approaches the centre of the sphere. The largest circle, *EFGH*, is obtained when the sphere is cut by a plane passing through the centre *O*. The circle then cuts the sphere

Fig. 185.

Fig. 186.

and its surface into two halves; it is called a *great circle.* The radius of a great circle is equal to the radius of the sphere.

Every pair of great circles intersect along the diameter of the sphere (*AB* in Fig. 186) which also serves as the diameter of each of the intersecting circles.

An infinity of great circles (meridians) can be drawn through two points of a spherical surface lying at the extremities of one and the same diameter. One and only one great circle can be drawn through two points not lying at the extremities of one diameter.

Fig. 187.

The shortest distance between two points on a spherical surface is an arc (smaller than a semicircle) of the great circle drawn through these points.

The surface area of a sphere is equal to four times the area of a great circle of the sphere:

$$S = 4\pi R^2$$

where $R$ is the radius of the sphere.

The volume of a sphere is equal to the volume of a pyramid whose base has the same area as the surface of the sphere, and the altitude is the radius of the sphere:

$$V = \frac{1}{3} RS = \frac{4}{3}\pi R^3$$

The volume of a sphere is one and a half times less than the volume of a cylinder circumscribed about it (Fig. 187), while the surface area of a sphere is $1\frac{1}{2}$ times less than the

total surface area of the cylinder (*Archimedes theorem*):

$$S=\frac{2}{3}S_1,$$

$$V=\frac{2}{3}V_1$$

where $S_1$ is the total area and $V_1$ is the volume of the cylinder shown in Fig. 187.

### 169. Spherical Polygons

A *spherical polygon* is a figure consisting of a closed series of arcs of great circles; no arc must exceed a half of a great circle. Figure 188 shows a spherical pentagon.

The arcs $AB$, $BC$ and so on are the *sides* of a spherical polygon; the points $A$, $B$, $C$, etc. are the *vertices*.

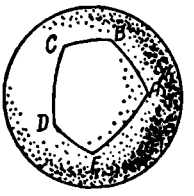


Fig. 188

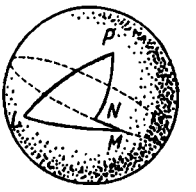


Fig. 189.

A spherical polygon is convex if its entire contour lies on *one* of the two hemispheres formed by the great circle containing one of the sides. The polygon $ABCDE$ of Fig. 188 is a convex polygon. The polygon $LMNP$ of Fig. 189 is not convex, since its contour lies in both hemispheres formed by the great circle of the side $NM$ (and also of side $NP$).

*Note.* Elementary geometry studies only simple spherical polygons, i.e., polygons whose contours have no self-intersections. Every simple polygon partitions a hemisphere into two regions, one of which can be taken as the interior region, the other as the exterior region. If the areas of the regions are not equal, the area of the smaller one is usually taken as the interior.

The interior angle of a spherical polygon, say angle $ABC$ which in Fig. 190 is denoted by $\beta$, is measured by the plane angle $A'BC'$ formed by the rays $BA'$, $BC'$ touching the

sides $BA$, $BC$. In place of the plane angle $A'BC'$ we can take the dihedral angle measured by it, the edge of which is the radius $OB$, and the faces are the planes $OBA'$, $OBC'$ of the great circles $BA$, $BC$.

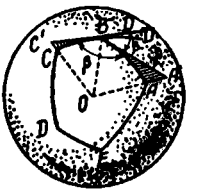

Fig. 190.

In the same way, the *exterior angle* of a spherical polygon, say angle $D''BA$ denoted in Fig. 190 by $\beta'$ is measured by the plane angle $D'BA'$ or by the corresponding dihedral angle. The sum of an interior angle and an exterior angle at one vertex is equal to 180°, or $\pi$ radians.

A plane polygon has at least three sides. A spherical polygon can have two. Fig. 191 depicts a spherical lune (two-sided polygon). The interior angles $\alpha$, $\beta$ of a lune are equal.

The area of a lune, the interior angle of which contains $\alpha$ radians, is given by the formula

$$S = 2R^2\alpha$$

where $R$ is the radius of the sphere.

**Example.** A lune whose interior angle is a right angle (quarter of a sphere) has area $2R^2 \cdot \frac{\pi}{2} = \pi R^2$, which is the same as that of a great circle (cf. Sec. 151).

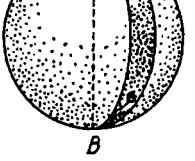

Fig. 191.

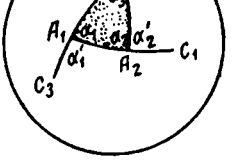

Fig 192.

In a spherical triangle the sum of the interior angles is always greater than 180°; the area of the triangle is proportional to the excess of this sum over 180°. Namely, if the interior angles contain $\alpha_1$, $\alpha_2$, $\alpha_3$ radians (Fig. 192), then

$$S = R^2(\alpha_1 + \alpha_2 + \alpha_3 - \pi) \qquad (1)$$

The sum of the exterior angles of a spherical triangle is always less than 360°. If $\alpha_1'$, $\alpha_2'$, $\alpha_3'$ are the exterior angles of a triangle expressed in radian measure, then

$$S = R^2 [2\pi - (\alpha_1' + \alpha_2' + \alpha_3')] \quad (2)$$

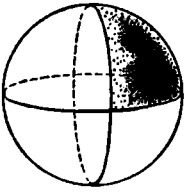

Fig. 193.

This formula can be extended to any spherical polygon. Namely

$$S = R^2 [2\pi - (\alpha_1' + \alpha_2' + \ldots + \alpha_n')]$$

That is, the ratio of the area of a spherical polygon to the square of the radius of the sphere is equal to the amount by which $2\pi$ is greater than the sum of the exterior angles.

**Example.** Let us consider the spherical triangle formed by three mutually perpendicular great circles (Fig. 193). The sum of the interior angles is $\frac{3\pi}{2}$. By formula (1) we get

$$S = \frac{1}{2}\pi R^2$$

The same result is obtained if we note that the given triangle constitutes $\frac{1}{8}$ of the sphere (cf. Sec. 168).

The sum of the exterior angles of the given triangle is also equal to $\frac{3\pi}{2}$. By formula (2) we again find $S = \frac{1}{2}\pi R^2$.

### 170. Parts of a Sphere

The portion of a sphere cut off by a plane (*ABCD* in Fig. 194) is called a *spherical segment of one base.*

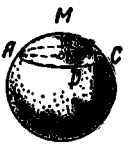


Fig. 194.

The *base* of a spherical segment is the circle *ABCD*. The altitude of a spherical segment is the line *NM*, or the perpendicular distance between the centre *N* of the base and the point of intersection with the surface of the sphere. The point *M* is called the *vertex* of the spherical segment.

The *curved surface area* of a spherical segment of one base is equal to the product of the altitude by the perime-

ter of a great circle of the sphere:

$$S = 2\pi Rh$$

where $R$ is the radius of the sphere and $h = MN$.

The *volume of a segment of one base* is

$$V = \pi h^2 \left(R - \frac{1}{3}h\right) \quad \text{or} \quad V = \frac{1}{6}\pi h\,(h^2 + 3r^2)$$

where $r$ is the radius of the base of the segment.

The portion of a sphere contained between two parallel

Fig. 195.

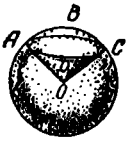


Fig. 196.

secant planes ($ABC$ and $DEF$ in Fig. 195) is called a *spherical segment of two bases*. The curved surface area of a spherical segment of two bases is called a *zone*. The circles $ACB$ and $DFE$ are the bases of the zone. The distance $NO$ between the bases is the *altitude* of the spherical segment of two bases (and zone).

The *curved surface area* of a spherical segment of two bases is equal to the product of the altitude $h = NO$ and the perimeter of a great circle of the sphere:

$$S = 2\pi Rh$$

The *volume of a spherical segment of two bases* is given by the formula

$$V = \frac{1}{6}\pi h^3 + \frac{1}{2}\pi\,(r_1^2 + r_2^2)\,h$$

where $r_1$ and $r_2$ are the radii of the bases.

That portion of a sphere bounded by the surface of a spherical segment of one base ($AC$ in Fig. 196) and a conical surface ($OABCD$), the base of which is the base of the segment ($ABCD$) and the vertex of which is the centre of the sphere is called a *spherical sector*.

The surface area of a *spherical sector* is made up of the surface area of a spherical segment of one base and the area of a cone.

The *volume of a spherical sector* is equal to the volume of a pyramid, the base of which has the same area as that part of the spherical surface cut out by the sector, and the altitude is equal to the radius of the sphere:

$$V = \frac{1}{3} RS = \frac{2}{3} \pi R^2 h$$

where $h$ is the altitude of the spherical segment of one base belonging to the spherical sector.

### 171. Tangent Plane to a Sphere, a Cylinder and a Cone

In practical situations, it is often possible to replace a small arc, $AB$, of a curve (say a circle) by a small segment $AT$ of the straight line that is tangent to the arc $AB$ at $A$ (Fig. 197). The error is usually insignificant. For instance, we say that we go from one point (on the earth's surface) to another along a straight line, whereas actually we move along the arc of a great circle drawn on the surface of the terrestrial sphere.

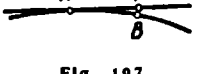

Fig. 197.

In the same way, a small part of a *curved surface* (say the surface of a sphere) can be replaced approximately by a small piece of the *tangent plane*, which is a plane that differs but slightly (almost imperceptibly) from a small portion of the curved surface. This was why for millenia people considered the surface of the earth to be flat (plane).

An exact definition of a tangent plane may be given in full accordance with the earlier given exact definition of a tangent line (see Sec. 151). There we considered two points $A$ and $B$ of a curve (say a circle); one of them was made to approach the other, and we noted that in the process the straight line $AB$ approached a certain limiting position. Now let us take three points $A$, $B$, $C$ (Fig. 198) on a surface (say of a sphere). Pass a cutting plane $P$ through them. Now let the two points $B$ and $C$ approach $A$ in two different directions. Then the $P$ plane will approach a certain limiting position $Q$ irrespective of where the points $B$ and $C$ were

taken and what their mode of approach to $A$ was. The $Q$ plane is called the tangent plane (at point $A$).*

The tangent plane to a surface at a point $A$ is the plane approached without limit by a secant plane passing through three points of the surface, $A$, $B$, $C$, when the points $B$ and $C$ approach $A$ from different directions. It may happen that the surface does not have any tangent plane at the point $A$. For example, a conical surface does not have a tangent plane at the vertex of the cone.

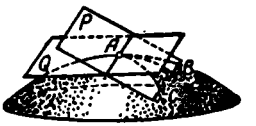

Fig. 198.

A plane ($Q$ in Fig. 199) that is *tangent to a spherical surface* is perpendicular to the radius $OA$ terminating at the point of tangency. A plane that is tangent to a spherical surface has only one point in common with the surface.

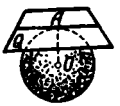

Fig. 199.

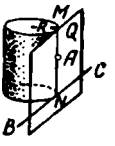

Fig. 200.

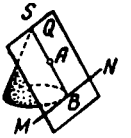

Fig. 201.

This property is ordinarily taken as a definition of a tangent plane to a sphere. However, it is not at all valid in the case of other surfaces, say the surfaces of a cylinder and cone, but the definition given earlier is applicable to these surfaces as well.

The plane $Q$ (Fig. 200) is a *tangent plane to the surface of a right circular cylinder* at the point $A$ and passes through the generator $MN$ which passes through $A$ and through the

---

* The requirement that points $B$ and $C$ approach $A$ in different directions is essential. If, for example, two travelers move towards the North Pole along one and the same meridian or along two meridians that are continuations of one another, then the plane passing through the pole, $A$, and the points $B$ and $C$ at which the travelers are located will always coincide with the plane of the meridian and, hence, will not approach the tangent plane, which is to say that it will always be the same cutting plane. This requirement may be stated in strict fashion as follows: lines tangent to the arcs $AC$ and $AB$ at their point $A$ of intersection must be distinct straight lines.

tangent line $BC$ to the base circle at point $N$, which belongs to the generator $MN$. The tangent plane to the surface of a right circular cylinder is separated from all points of the axis of the cylinder by the radius $R$ of the base of the cylinder.

The plane $Q$ (Fig. 201), which is the *tangent plane to the surface of a right circular cone at the point A* ($A$ does not coincide with the vertex $S$) passes through the generator $SB$, which passes through point $A$, and through the tangent line $MN$ to the base circle at point $B$.

A cylinder is said to be an *inscribed cylinder of a prism* if the lateral faces of the prism are tangent to the cylinder and the bases of prism and cylinder are coplanar (in the same plane). A cylinder is a *circumscribed cylinder of a prism* if the lateral edges of the prism are generating elements of the lateral surface of the cylinder and the bases of prism and cylinder are coplanar.

The definitions are the same for an *inscribed cone of a pyramid* and *a circumscribed cone of a pyramid.*

### 172. Solid Angles

A *solid angle* is a portion of space within one nappe of a conical surface (see Sec. 166) with closed directrix. Like an angle between two straight lines in a plane, a solid angle extends without bound (an infinite funnel). A polyhedral angle (Sec. 163) is a special case of a solid angle (a pyramidal surface is a special case of a conical surface).

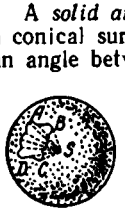

Fig. 202.

Just as an angle between two straight lines is measured by the arc of a circle, a solid angle is measured by a portion of the surface of a sphere. To see this, from vertex $S$ of a solid angle, draw a spherical surface of arbitrary radius. The surface of the solid angle will cut out of this surface a portion: $ABCD$ in Fig. 202. The area of this portion will vary with the radius of the sphere but will always constitute one and the same fraction of the surface area of the sphere. For this reason, for the measure of a solid angle we could take the ratio of the area $ABCD$ to the area of the spherical surface, in the same way that the angle between two straight lines might be measured by the ratio of the arc subtended

by them (with centre at the vertex of the angle) to the perimeter of a circle of the same radius (an angle of half a rotation, one-fourth rotation, etc.). However, the accepted measure for a solid angle is the ratio of the area $ABCD$ to the area of a square constructed on the radius of the sphere (it is expressed by the quantity $R^2$, which is proportional to the surface area $4\pi R^2$ of the sphere). This measure of solid angles is similar to the radian measure of angles between straight lines (see Sec. 180).

Thus, for the measure $\alpha$ of a solid angle with vertex $S$ we take the ratio of the area cut out by a solid angle on the surface of a sphere of arbitrary radius with centre $S$ to the square of the radius of the sphere:

$$\alpha = \frac{\text{area } ABCD}{R^2}$$

**Example 1.** A solid angle formed by three mutually perpendicular planes (for instance, by two sides and the bottom of a rectangular box) is equal to $\frac{\pi}{2}$. Indeed, if a spherical surface is described from vertex $S$ of such a solid angle, $\frac{1}{8}$ of it will be cut out on the surface of the sphere (Fig. 203) since three mutually perpendicular planes will cut it into 8 equal parts (imagine a portion of the surface of a globe cut out by two mutually perpendicular planes passing through meridians and by a plane passing through the equator); hence, the area of this portion of the surface is equal to $4\pi R^2 : 8 = \frac{\pi R^2}{2}$ and its ratio to $R^2$ is equal to $\frac{\pi}{2}$.

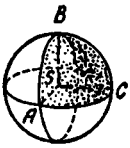

Fig. 203.

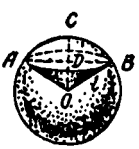

Fig. 204.

**Example 2.** Find the solid angle at the vertex of a cone whose altitude is equal to the radius of the base.

Draw from the vertex of the cone a sphere of radius equal to the generator $l$ of the cone (Fig. 204). The altitude $OD$ of the cone may be expressed in terms of $l$: $OD = \frac{l\sqrt{2}}{2}$; the altitude $CD$ of the spherical segment $ABC$ is then $l - \frac{l\sqrt{2}}{2}$; the spherical area cut out by the solid angle is the curved

*Continued*

| ° | ′ | log sin | d | log tan | cd | log cot | d | log cos | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 0 | 9.2806 | 64 | 9.2887 | 66 | 0 7113 | 2 | 9.9919 | 0 | 79 |
| | 10 | 9.2870 | 64 | 9.2953 | 67 | 0 7047 | 3 | 9.9917 | 50 | |
| | 20 | 9.2934 | 63 | 9.3020 | 65 | 0.6980 | 2 | 9.9914 | 40 | |
| | 30 | 9.2997 | 61 | 9.3085 | 64 | 0.6915 | 3 | 9.9912 | 30 | |
| | 40 | 9.3058 | 61 | 9.3149 | 63 | 0.6851 | 2 | 9.9909 | 20 | |
| | 50 | 9.3119 | 60 | 9.3212 | 63 | 0.6788 | 3 | 9.9907 | 10 | |
| 12 | 0 | 9.3179 | 59 | 9.3275 | 61 | 0.6725 | 3 | 9.9904 | 0 | 78 |
| | 10 | 9.3238 | 58 | 9.3336 | 61 | 0.6664 | 2 | 9.9901 | 50 | |
| | 20 | 9.3296 | 57 | 9.3397 | 61 | 0.6603 | 3 | 9.9899 | 40 | |
| | 30 | 9.3353 | 57 | 9.3458 | 59 | 0.6542 | 3 | 9.9896 | 30 | |
| | 40 | 9.3410 | 56 | 9 3517 | 59 | 0.6483 | 3 | 9.9893 | 20 | |
| | 50 | 9.3466 | 55 | 9.3576 | 58 | 0 6424 | 3 | 9.9890 | 10 | |
| 13 | 0 | 9.3521 | 54 | 9.3634 | 57 | 0.6366 | 3 | 9.9887 | 0 | 77 |
| | 10 | 9 3575 | 54 | 9.3691 | 57 | 0.6309 | 3 | 9.9884 | 50 | |
| | 20 | 9.3629 | 53 | 9.3748 | 56 | 0.6252 | 3 | 9.9881 | 40 | |
| | 30 | 9.3682 | 52 | 9.3804 | 55 | 0.6196 | 3 | 9.9878 | 30 | |
| | 40 | 9.3734 | 52 | 9.3859 | 55 | 0.6141 | 3 | 9.9875 | 20 | |
| | 50 | 9.3786 | 51 | 9.3914 | 54 | 0.6086 | 3 | 9.9872 | 10 | |
| 14 | 0 | 9.3837 | 50 | 9.3968 | 53 | 0.6032 | 3 | 9.9869 | 0 | 76 |
| | 10 | 9.3887 | 50 | 9.4021 | 53 | 0.5979 | 3 | 9.9866 | 50 | |
| | 20 | 9.3937 | 49 | 9.4074 | 53 | 0.5926 | 4 | 9.9863 | 40 | |
| | 30 | 9 3986 | 49 | 9.4127 | 51 | 0.5873 | 3 | 9.9859 | 30 | |
| | 40 | 9.4035 | 48 | 9.4178 | 52 | 0.5822 | 3 | 9.9856 | 20 | |
| | 50 | 9.4083 | 47 | 9.4230 | 51 | 0.5770 | 4 | 9.9853 | 10 | |
| 15 | 0 | 9.4130 | 47 | 9.4281 | 50 | 0.5719 | 3 | 9.9849 | 0 | 75 |
| | 10 | 9.4177 | 46 | 9.4331 | 50 | 0.5669 | 3 | 9.9846 | 50 | |
| | 20 | 9.4223 | 46 | 9.4381 | 49 | 0.5619 | 4 | 9.9843 | 40 | |
| | 30 | 9.4269 | 45 | 9.4430 | 49 | 0.5570 | 3 | 9.9839 | 30 | |
| | 40 | 9.4314 | 45 | 9.4479 | 48 | 0.5521 | 4 | 9.9836 | 20 | |
| | 50 | 9.4359 | 44 | 9.4527 | 48 | 0.5473 | 4 | 9.9832 | 10 | |
| 16 | 0 | 9.4403 | 44 | 9.4575 | 47 | 0.5425 | 3 | 9.9828 | 0 | 74 |
| | 10 | 9.4447 | 44 | 9 4622 | 47 | 0.5378 | 4 | 9.9825 | 50 | |
| | 20 | 9.4491 | 42 | 9.4669 | 47 | 0.5331 | 4 | 9.9821 | 40 | |
| | 30 | 9.4533 | 43 | 9.4716 | 46 | 0.5284 | 3 | 9.9817 | 30 | |
| | 40 | 9 4576 | 42 | 9.4762 | 46 | 0.5238 | 4 | 9.9814 | 20 | |
| | 50 | 9 4618 | 41 | 9 4808 | 45 | 0.5192 | 4 | 9.9810 | 10 | |
| | | log cos | d | log cot | cd | log tan | d | log sin | ′ | ° |

surface area of this spherical segment and is equal (see Sec. 170) to

$$2\pi l \cdot CD = 2\pi l^2 \left(1 - \frac{\sqrt{2}}{2}\right)$$

Hence the solid angle is

$$2\pi \left(1 - \frac{\sqrt{2}}{2}\right)$$

The unit solid angle is a solid angle that cuts out of a sphere (with centre at the vertex of the angle) an area equal to that of the square constructed on the radius. The unit solid angle is called a *steradian.*

### 173. Regular Polyhedrons

A *polyhedron is regular* if all its faces are congruent regular polygons and the same number of faces meet at each vertex.

Fig. 205.

Fig. 206.

Fig. 207.

Fig. 208.

Fig. 209.

In contrast to the fact that there are an infinity of nonsimilar regular polygons, there is only a limited number of regular polyhedrons that are not similar. There can only be five regular convex polyhedrons (and another four regular concave polyhedrons). These five regular convex polyhedrons are: the *regular tetrahedron* or, simply, *tetrahedron* (Fig. 205); the *hexahedron* (or *cube*, Fig. 206); the *octahedron* (Fig. 207); the *dodecahedron* (Fig. 208); the *icosahedron* (Fig. 209).

The following table gives the number of vertices and edges, and also the surface areas and volumes expressed in terms of an edge $a$ for the regular convex polyhedrons.

| | Number of sides of each face | Number of edges at each vertex | Number of faces | Number of vertices | Number of edges | Surface | Volume |
|---|---|---|---|---|---|---|---|
| Tetrahedroon | 3 | 3 | 4 | 4 | 6 | $1.73\ a^2$ | $0.12\ a^3$ |
| Hexahedron (cube) | 4 | 3 | 6 | 8 | 12 | $6.00\ a^2$ | $a^3$ |
| Octahedron | 3 | 4 | 8 | 6 | 12 | $3.46\ a^2$ | $0.47\ a^3$ |
| Dodecahedron | 5 | 3 | 12 | 20 | 30 | $20.64\ a^2$ | $7.66\ a^3$ |
| Icosahedron | 3 | 5 | 20 | 12 | 30 | $8.66\ a^2$ | $2.18\ a^3$ |

A sphere can be inscribed in, and circumscribed about, every regular polyhedron.

### 174. Symmetry

The word *symmetry* comes from the Greek meaning balanced proportions. In the broad sense of the word, symmetry is to be understood as any regularity in the inner structure of a body or figure. The study of various kinds of symmetry represents an extensive and important branch of geometry that is closely bound up with many fields of natural science and engineering, ranging from textiles to intricate problems in the structure of matter.

There are three elementary types of symmetry:

1. *Mirror symmetry*, which is commonplace in our everyday lives. As the name implies, mirror symmetry relates an object to its image in a plane mirror. The geometric definition of mirror symmetry is this: *a figure* (Fig. 210) is called *symmetric with respect to a plane P* (mirror plane, plane of symmetry) if each point $E$ of the figure is associated with an identical point $E'$ of the same figure such

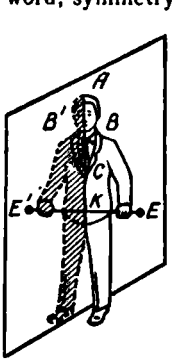


Fig. 210.

that the line $EE'$ is perpendicular to the plane $P$ and is cut in half by it.

We say that one figure (or solid) is mirror-symmetric with respect to another if together they form a mirror-symmetric figure (or solid). In Fig. 210, the line $ABC$ is symmetric with respect to the line $AB'C$; the right hand is symmetric to the left.

It is important to note that two symmetric bodies cannot, generally speaking, be "inserted one into the other"; to put it differently, one of the bodies cannot take the place of the other. This is clearly illustrated by the fact that a left-hand glove does not fit the right hand.

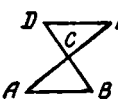

Fig. 211.

Symmetric figures are essentially different despite their many similarities. This is cogently demonstrated by putting a written message in front of a mirror and trying to read the text in the mirror.

Symmetric objects cannot therefore be called equal in the narrow sense of the word. They are called *mirror-equivalent*. Generally, mirror-equivalent bodies (or figures) are such that when they are displaced (translated) in some appropriate fashion they can form two halves of a mirror-symmetric body (or figure).

2. *Central symmetry.* A figure (or solid) is said to be symmetric about a centre $C$ if every point $E$ of the figure (solid) is associated with a point $A$ of the same figure (solid) such that the line $EA$ passes through the point $C$ which bisects it (Fig. 211). The figure $ABCDE$ composed of two triangles $ABC$ and $EDC$ (Fig. 211) whose sides are equal in pairs and are extensions of each other has a *centre of symmetry* ($C$). Between corresponding pairs of points lie equal line-segments; the corresponding angles of the two halves of a solid possessing central symmetry are also equal. Generally speaking, however, the two halves of a solid having central symmetry cannot be interchanged; this is the same for the two halves of a solid possessing mirror symmetry. What is more, one of the halves of a solid with central symmetry may (via a rotation through 180° about any axis passing through the centre of symmetry) be carried to a mirror-symmetric position relative to the other (with respect to the plane perpendicular to the axis of rotation). For this reason, the two halves of a solid having central symmetry are mirror-equivalent (see above).

**Example.** If edges $SA$, $SB$, $SC$, ... of the pyramid

*SABCDE* (Fig. 212) are produced to distances equal to the lengths of these edges and in the opposite direction from the vertex, then the two pyramids *SABCDE* and *Sabcde* jointly form a solid symmetric about the centre *S*.

If the pyramid *SABCDE* in Fig. 212 is hollow and without a "bottom" *ABCDE* (a pyramidal funnel), then, by turning it inside out, we get a solid into which we can insert

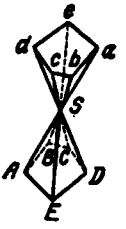


Fig. 212.

Fig. 213.

pyramid *Sabcde*; in the general case, it is not possible to bring these two solids to coincidence without turning one inside out, so that in the general case *SABCDE* and *Sabcde* are not equal but only mirror-equivalent. Equality is possible in exceptional cases (say if *SABCDE* is a regular pyramid).

3. *Rotational symmetry.* A solid (or figure) has rotational symmetry if when rotated through an angle $\frac{360°}{n}$ ($n$ an integer) about a straight line *AB* (axis of symmetry) it coincides completely with its original position. If the number $n$ is equal to 2, 3, 4 etc., the axis of symmetry is termed a *two-fold, three-fold, four-fold,* etc. *axis.*

**Example.** Cut a circle into three sectors with central angles 120° each (Fig. 213). Superimpose the sectors without turning them over and cut out a figure *a* of arbitrary shape. Now if the sectors are spread out in the original position, we get a figure (a circle with three odd-shaped holes) which possesses a three-fold symmetry axis. This axis is perpendicular to the plane of the drawing. A rotation through 120° brings the figure to full coincidence with its original position.

In a narrower sense, an axis of symmetry is a two-fold symmetry axis, and one speaks of "axial symmetry", which may be defined thus: a figure (or body) has axial symmetry with respect to some axis if every point *E* of it is associated

with another point, $F$, of the same figure such that the line-segment $EF$ is perpendicular to the axis, intersects it and is bisected by the point of intersection. The earlier considered pair of triangles (see Fig. 211) has axial symmetry in addition to central symmetry. Its axis of symmetry passes through point $C$ perpendicular to the plane of the drawing.

The following are some instances of the symmetry types discussed above.

A *sphere* has central, mirror and axial symmetry. The centre of symmetry lies at the centre of the sphere, the plane of any great circle is a plane of symmetry; and any diameter of the sphere is an axis of symmetry. The sphere has $n$-fold symmetry, where $n$ is any integer.

A *right circular cone* has $n$-fold axial symmetry; the axis of symmetry is the axis of the cone.

A *regular pentagonal prism* has a plane of symmetry passing parallel to the bases at equal distances from them, and the axis of symmetry is five-fold and coincides with the axis of the prism. A plane bisecting one of the dihedral angles formed by the lateral faces can also serve as the plane of symmetry.

### 176. Symmetry of Plane Figures

1. *Mirror-axial symmetry.* If a plane figure ($ABCDE$ in Fig. 214) is symmetric about a plane $P$ (this is possible if

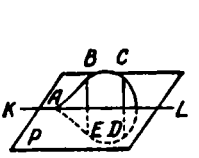


Fig. 214.

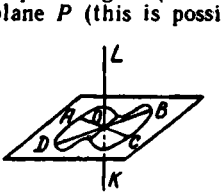


Fig. 215.

planes $ABCDE$ and $P$ are mutually perpendicular), then the straight line $KL$ in which the planes intersect is the axis of symmetry (two-fold) of the figure $ABCDE$. Conversely, if the plane figure $ABCDE$ has $KL$ as the axis of symmetry in that plane, then this figure is symmetric about the plane $P$ passed through $KL$ perpendicular to the plane of the figure. For this reason, the axis $KL$ may also be called the *mirror line* of the plane figure $ABCDE$.

Two mirror-symmetric plane figures can always be superimposed on each other. However, to do this, it is necessary to take one (or both) outside their common plane.

2. *Central symmetry.* If a plane figure (*ABCD* in Fig. 215) has a two-fold axis of symmetry perpendicular to the plane of the figure (the straight line *KL* in Fig. 215), then point *O* in which *KL* intersects the plane of the figure is the centre of symmetry of the figure *ABCD*. Conversely, if the plane figure *ABCD* has centre of symmetry *O* (it invariably lies in the plane of the figure), then this figure has a two-fold symmetry axis which passes through *O* perpendicular to the plane of the figure. It is thus always possible to superimpose two central-symmetric plane figures without taking them outside their common plane. To do this, it suffices to turn one of them through an angle of 180° about the centre of symmetry.

As in the case of mirror symmetry, a plane figure in the case of central symmetry invariably has a two-fold symmetry axis, but in the former case the axis lies in the plane of the figure, while in the latter case it is perpendicular to this plane.

For this reason, in plane geometry we have axial symmetry only in the former case.

### 176. Similarity of Solids

The *similarity* of solids may be defined in the same way as the similarity of plane figures (see Sec. 150). Two solids are *similar* if one of them is obtained from the other by increasing or decreasing all (linear) dimensions in the same ratio. A machine and a model of the machine are similar bodies. Two solids (or figures) are *mirror-similar* if one of them is similar to the mirror image of the other (see Sec. 174). Thus, for example, the negative of a photograph of a portrait is mirror-similar to the portrait. Two shoes of different sizes, but of the same design, one of the left foot, the other of the right foot, are mirror-similar.

In similar and mirror-similar figures all corresponding angles (plane and dihedral) are equal. In similar solids, polyhedral and solid angles are equal, in mirror-similar solids they are mirror-equivalent.

If in two tetrahedrons (i. e., two triangular pyramids) the corresponding edges are proportional (or, what is the same,

the corresponding faces are similar), then they are similar or mirror-similar, so that, for instance, if the edges of one are twice the edges of the other, then the altitudes of one are twice the altitudes of the other and the radius of a circumscribed sphere of one is twice the radius of the other, etc.

This theorem no longer holds for polyhedrons with a larger number of faces. For instance, suppose 12 equal rods are connected so as to form the edges of a cube. If the joints at the vertices are of the hinged type, then, without extending the rods, we can alter the shape of the cube to obtain a parallelepiped $P$. A parallelepiped $P_1$ which is similar to $P$ will not be similar or mirror-similar to the cube, although its edges are proportional to the edges of the cube; this does not occur in the case of a tetrahedron made of 6 rods since it preserves its shape even if we have hinge joints.

Thus, generally speaking, proportionality of all edges is not sufficient for bodies to be similar (or mirror-similar).

Two prisms or two pyramids are similar or mirror-similar if the base and one of the lateral faces of one are similar to the corresponding base and lateral edge of the other and, besides, if the dihedral angles formed in both prisms (pyramids) by the indicated faces are equal. Two regular prisms or pyramids with the same number of faces are similar if the radii of their bases are proportional to their altitudes. Two right circular cylinders or cones are similar if the radii of their bases are proportional to their altitudes.

In similar bodies, the areas of all corresponding plane and curved surfaces are proportional to the squares of arbitrary corresponding line-segments (that is, the ratio of the areas is equal to the square of the ratio of similitude).

The volumes of similar bodies and also the volumes of arbitrary corresponding parts of them are proportional to the cubes of arbitrary corresponding line-segments (that is, the ratio of volumes is equal to the cube of the ratio of similitude).

These last two properties enable us to simplify certain computations very substantially.

**Example 1.** A total of 6.5 kg of drying oil is required in the painting of a hemispherical dome of diameter 5 metres. How much drying oil is needed to paint a dome of diameter 8 metres?

Any two hemispheres are similar bodies. Hence, their surface areas and the amount of drying oil needed to paint them are proportional to the squares of the diameters.

Denoting by $x$ the desired quantity of drying oil, we have

$$\frac{x}{6.5}=\left(\frac{8}{5}\right)^3, \quad x=6.5\cdot\left(\frac{8}{5}\right)^3\approx 16.6 \text{ kg}$$

**Example 2.** A tin can 11 cm in height and 8 cm in diameter contains 0.5 kg of jam. What size can (same shape) will hold 1 kg of jam?

Denoting the altitude by $h$ and the diameter of the base of the can by $d$, we have $\left(\frac{h}{11}\right)^3=\frac{1}{0.5}=2$, whence

$$h=11\sqrt[3]{2}\approx 14 \text{ cm}$$

In the same way, $d=8\sqrt[3]{2}\approx 10$ cm.

### 177. Volumes and Areas of Solids

The following symbols are used: $V$, volume; $S$, area of base; $S_{lat}$, lateral area; $P$, total area; $h$, altitude; $a$, $b$, $c$, the dimensions of a rectangular parallelepiped; $A$, slant height of regular pyramid and regular truncated pyramid (parallel bases); $l$, generator of cone; $p$, perimeter of base; $r$, radius of base; $d$, diameter of base; $R$, radius of sphere; $D$, diameter of sphere.

*Prism*, right or oblique, and *parallelepiped*:

$$V=Sh$$

Right prism:

$$S_{lat}=ph$$

Rectangular parallelepiped:

$$V=abc; \quad P=2(ab+bc+ac)$$

*Cube:*

$$V=a^3, \quad P=6a^2$$

Regular or irregular *pyramid*:

$$V=\frac{1}{3}Sh$$

Regular pyramid:

$$S_{lat}=\frac{1}{2}pA$$

Truncated pyramid (parallel bases), regular or irregular:

$$V = \frac{1}{3}(S_1 + \sqrt{S_1 S_2} + S_2)\, h$$

Truncated pyramid (parallel bases), regular:

$$S_{lat} = \frac{1}{2}(p_1 + p_2)\, A$$

Circular *cylinder* (right or oblique):

$$V = Sh = \pi r^2 H = \frac{1}{4}\pi d^2 h$$

Right circular cylinder:

$$S_{lat} = 2\pi rh = \pi dh$$

Circular *cone* (right or oblique):

$$V = \frac{1}{3}Sh = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi d^2 h$$

Right circular cone:

$$S_{lat} = \frac{1}{2}pl = \pi rl = \frac{1}{2}\pi dl$$

Truncated (parallel bases) circular cone (right or oblique):

$$V = \frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2) = \frac{1}{12}\pi h(d_1^2 + d_1 d_2 + d_2^2)$$

Truncated (parallel bases) right circular cone:

$$S_{lat} = \pi(r_1 + r_2)\, l = \frac{1}{2}\pi(d_1 + d_2)\, l$$

*Sphere:*

$$V = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3, \quad P = 4\pi R^2 = \pi D^2$$

*Hemisphere:*

$$V = \frac{2}{3}\pi R^3 = \frac{1}{12}\pi D^3, \quad S = \pi R^2 = \frac{1}{4}\pi D^2,$$

$$S_{lat} = 2\pi R^2 = \frac{1}{2}\pi D^2, \quad P = 3\pi R^2 = \frac{3}{4}\pi D^2$$

*Spherical segment of one base:*

$$V = \pi h^2\left(R - \frac{1}{3}h\right) = \frac{\pi h}{6}(h^2 + 3r^2),$$

$$S_{lat} = 2\pi Rh = \pi(r^2 + h^2), \quad P = \pi(2r^2 + h^2)$$

*Spherical segment of two bases:*

$$V = \frac{1}{6}\pi h^3 + \frac{1}{2}\pi (r_1^2 + r_2^2)\, h,$$

$$S_{lat} = 2\pi R h'$$

*Spherical sector:*

$$V = \frac{2}{3}\pi R^2 h'$$

where $h'$ is the altitude of the segment contained in the sector.

*Hollow sphere:*

$$V = \frac{4}{3}\pi (R_1^3 - R_2^3) = \frac{\pi}{6}(D_1^3 - D_2^3),$$

$$P = 4\pi (R_1^2 + R_2^2) = \pi (D_1^2 + D_2^2)$$

where $R_1$ and $R_2$ are the radii of the inner and outer spherical surfaces, respectively.

# TRIGONOMETRY

## 178. The Subject of Trigonometry

The word "trigonometry" is derived from two Greek words, "trigonon", triangle, and "metria", measure. The basic task of trigonometry is the *solution of triangles*, finding unknown quantities of a triangle from given values of other of its quantities. Such, for example, is the problem of computing the angles of a triangle from given sides, computing the sides of a triangle from the area and two angles, etc. Since any computational problem of geometry may be reduced to the solution of triangles, trigonometry finds applications in the entire field of plane and solid geometry, and is extensively used in many areas of natural science and engineering.

The theory of the solution of spherical triangles (Sec. 169) is called *spherical trigonometry*, in contrast to which the solution of ordinary triangles is termed *plane trigonometry*.

The angles of an arbitrary triangle cannot be connected with its sides by means of algebraic relations. For this reason, trigonometry introduces new quantities in addition to the angles themselves: these are the so-called *trigonometric functions* (defined in Sec. 182), which can be connected with the sides of a triangle by simple algebraic relations. On the other hand, the value of a trigonometric function can be computed from a given angle, and conversely. True, these computations are arduous and unwieldy, but the work has been done once and for all and recorded in tables.

The value of each trigonometric function varies with the angle to which it corresponds; in other words, a trigonometric quantity is a function of the angle (Sec. 207), whence the name trigonometric functions.

The various trigonometric functions are interrelated in a variety of important ways. These relationships can be used to reduce and simplify computations. That part of trigonometry which deals with such relations is termed *analytical trigonometry* (or *goniometry*, the measurement of angles, from the Greek "gonia", angle, and "metria", measure).

### 179. Historical Survey of the Development of Trigonometry

Solving triangles originated in astronomy and for a long time trigonometry developed as a department of astronomy.

As far as we know, methods for solving (spherical) triangles were recorded for the first time by the Greek astronomer Hipparchus in the middle of the 2nd century B. C. (this work is not extant). The highest attainments of Greek trigonometry are due to the astronomer Ptolemy (2nd century B. C.), the creator of the geocentric system of the world which dominated science till the time of Copernicus.

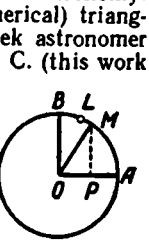

Fig. 216.

The Greek astronomers did not deal in sines, cosines and tangents. In place of tables of these quantities they used tables that permitted finding the chord of a circle from the subtended arc. Arcs were measured in degrees and minutes; chords too were measured in degrees (one degree being one sixtieth of the radius), minutes and seconds. The Greeks borrowed this sexagesimal scale from the Babylonians (see Sec. 21).

The tables compiled by Ptolemy contained the chords of all arcs for $\frac{1^\circ}{2}$ intervals * computed to within a second. Using interpolation, the chord of any arc may be found in them to the same degree of accuracy. To simplify interpolations, Ptolemy gave 1′ corrections. Ptolemy computed the tables on the basis of a theorem he discovered called the theorem of the diagonals of an inscribed quadrilateral (see Sec. 156).

Medieval astronomers of India also made considerable advances in trigonometry. Like the Greeks, they borrowed the Babylonian degree system of measuring angles. But the Hindus considered not the chords of arcs but the lines of the sines and consines (that is, the lines $PM$ and $OP$ for arc $AM$ in Fig. 216). They also considered the line $PA$, which later became known in Europe as the versed sine ("sinus versus").

An arc minute was taken as the unit of measure of lne-segments $MP$, $OP$, $PA$. Thus, the line of the sine of

* If we take the central angle intercepting half the arc at hand, then the chord will be twice the line of the sine of this angle. Ptolemy's table is therefore equivalent to a five-place table of sine values for $\frac{1^\circ}{4}$ intervals.

arc $AB=90°$ is $OB$, the radius of the circle; the arc $AL$, equal to the radius, contains $57°18'=3438'$ (rounded). Therefore, the sine of an arc of 90° was taken equal to 3438′.

The extant Hindu tables of sines (the most ancient of them was compiled in the 4th or 5th century) are not as exact as the Ptolemaic tables: they are compiled for angles differing by 3°45′, that is, by 1/24 part of the arc of a quadrant.

Trigonometry saw further development in the 9th to 14th centuries in the works of Arabic-writing authors. In the 10th century, the Bagdad scholar Mohammed of Bujan, known generally as Abû'l-Wefâ, added to the lines of sines and cosines the lines of tangents, cotangents, secants and cosecants. He gave them the same definitions that we find today in textbooks. He also established the basic relationships among these lines (see Sec. 191 for the appropriate formulas). It was in the hands of the famous Moslem scholar Nasir Eddin (or Nasr ed-din) of Tusi (1201—1274) that trigonometry became an independent scientific discipline. Nasir Eddin made a systematic study of all cases of the solution of plane and spherical triangles and gave new methods of solving them.

In the 12th century, a series of astronomical works were translated from the Arabic language into Latin, and Europeans thus first learned of trigonometry. * However, many of the achievements of the Arabic-language science, one of which was a work by Nasir Eddin, remained unknown to Europeans. Two hundred years later, the outstanding German astronomer of the 15th century Johann Müller (1436—1476) better known under the name Regiomontanus, rediscovered his theorems.

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* It was at this time that the Latin term "sinus" appeared, which meant "bay" or "inlet" or "bosom of a garment". This is a translation of the Arabic word "jaib" meaning the same thing. It is not known how this Arabic term originated. Some believe that it came from the Hindu (Sanskrit) "jiva" (the first meaning of which is bowstring; in geometry, it meant chord of an arc). But sine in Hindu terminology is designated by "ardha-jiva" which means "half chord".

The name "cosine" appeared only at the beginning of the 17th century as a contraction of the term complement sinus (sine of the complement), which indicated that the cosine of an angle $A$ is the sine of the complementary angle. The terms "tangent" and "secant" (which, translated from the Latin, mean "contacting", and "cutting") were introduced in 1583 by the German scholar Finck.

Regiomontanus compiled extensive tables of sines (for 1′intervals accurate to the seventh significant digit). He was the first to reject the sexagesimal division of the radius, and for the unit of measure of the sine line he took one ten-millionth part of the radius. Thus, sines were expressed as whole numbers (and not sexagesimal fractions). It was one step more to the introduction of decimal fractions. But this step required over 100 years (see Sec. 45).

Regiomontanus' tables were followed by other, still more detailed, tables. G. Rheticus (or Rhaeticus, 1514—1576), friend and collaborator of Copernicus, spent 30 years with hired computors in compiling tables that were finally completed and published in 1596 by his pupil Otho. The angles were given for every 10″ of arc and the radius was divided into 1,000,000,000,000,000 parts so that the sines had 15 correct digits!

Literal symbolism, which in algebra came in at the end of the 16th century, was established in trigonometry only in the middle of the 18th century thanks to the efforts of the great Euler (1707—1783), who gave trigonometry its modern aspect. The quantities $\sin x$, $\cos x$, etc. were regarded by him as functions (Sec. 179) of a number $x$, the radian measure of the appropriate angle. Euler assigned to the number $x$ all possible values: positive, negative and even complex. He also introduced the inverse trigonometric functions (Sec. 201).

### 180. Radian Measure of Angles

Besides the degree measure of angles (Sec. 142), trigonometry makes use of the so-called *radian* measure of angles. Here, the unit of measure is the *acute central angle* ($MON$ in Fig. 217) subtended in a circle by an arc whose length is equal to the radius of the circle ($\overset{\frown}{MN} = OM$). Such an angle is termed a *radian*. The value of this angle is not dependent on the radius of the circle and on the position of the arc $MN$ on the circumference. Since a semicircle is seen from the centre at an angle of 180°, and its length is equal to $\pi$ radii, the radian is $\pi$ times less

Fig. 217.

than a 180° angle, i.e , one radian is equal to $\frac{180}{\pi}$ degrees:

$$1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57^\circ.2958 \approx 57^\circ 17' 45''$$

Conversely, one degree is equal to $\frac{\pi}{180}$ radian.

$$1^\circ = \frac{\pi}{180} \text{ radian} \approx 0.017453 \text{ radian},$$

$$1' = \frac{\pi}{180 \cdot 60} \text{ radian} \approx 0.000291 \text{ radian},$$

$$1'' = \frac{\pi}{180 \cdot 60 \cdot 60} \text{ radian} \approx 0.000005 \text{ radian}$$

The radian measure of an angle ($AOB$ in Fig. 218) is the ratio of the angle to the radian ($MON$ in Fig. 218); but the ratio $\angle AOB : \angle MON$ is equal to the ratio of the arcs $\overset{\smile}{AB} : \overset{\smile}{MN}$, that is, the ratio of the arc $AB$ to the radius.

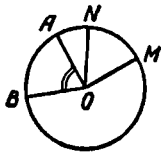

Fig. 218.

Thus, the radian measure of any angle $AOB$ is the ratio of the length of arc $AB$, described by an arbitrary radius from centre $O$ and contained between the sides of the angle, to the radius $OA$ of this arc.

The introduction of radian measure of angles permits casting many formulas in simpler forms. *

It will be useful to remember the following comparative table of degree and radian measure of some of the frequently

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* In many trigonometry texts it is stressed that in the radian measure of angles, the value of the angle is measured by a pure (abstract) number. The contrast thus created between the radian and degree measure of angles is not justifiable. In both systems (radian and degree) the angle is measured by a *unit of angle*. The fact that in one case (degree) the name is stated and in the other (radian) it is assumed, plays no role whatsoever.

The only reasonable meaning of the foregoing assertion is that the radian measure, which is expressed by the ratio of two lengths, is completely independent of any choice of unit of length. But neither is the degree measure dependent on this choice; what is more, it too is a ratio of two lengths: namely the length of the arc described from the vertex of the angle and intercepted by the arms of the angle to $\frac{1}{360}$ the part of the arc of a circle of the same radius. This ratio is in no way worse than the ratio of the same arc to its radius.

occurring angles:

| Angles in degrees | 360° | 180° | 90° | 60° | 45° | 30° |
|---|---|---|---|---|---|---|
| Angles in radians | $2\pi$ | $\pi$ | $\frac{\pi}{2}$ | $\frac{\pi}{3}$ | $\frac{\pi}{4}$ | $\frac{\pi}{6}$ |

### 181. Changing from Degrees to Radians and from Radians to Degrees

1. To find the radian measure of an angle from a given degree measure, multiply (see Sec. 180) the number of degrees by $\frac{\pi}{180} \approx 0.017453$, the number of minutes by $\frac{\pi}{180 \cdot 60}$ $\approx 0.000291$, and the number of seconds by $\frac{\pi}{180 \cdot 60 \cdot 60}$ $\approx 0.000005$, and add the products thus found.

**Example 1.** Find the radian measure of the angle 12°30′ to within four decimal places.

**Solution.** Multiply 12 by $\frac{\pi}{180}$, taking the fifth decimal place of the multiplier (because multiplication by 12 will increase the absolute error by a factor of about 10, cf. Sec. 54); $12 \cdot 0.01745 = 0.2094$.

Multiply 30 by $\frac{\pi}{180 \cdot 60}$ taking into account the sixth decimal place of the multiplier; $30 \cdot 0.000291 \approx 0.0087$. This yields $12°30' = 0.2094 + 0.0087 = 0.2181$.

The computations are simplified by using the table of Sec. 8, page 48. It gives results accurate to four decimal places. In the first column ("degrees") opposite 12 we find 0.2094; in the second last column ("minutes") opposite 30 we find 0.0087.

**Work:**

$$\begin{aligned} 12° &= 0.2094 \text{ (radian)} \\ 30' &= 0.0087 \\ \hline & 0.2181 \end{aligned}$$

**Example 2.** Find the radian measure of the angle 217°40′. Using the same table, we have

$$\begin{aligned} 200° &= 3.4907 \\ 17° &= 0.2967 \\ 40' &= 0.0116 \\ \hline & 3.7990 \end{aligned}$$

2. To find the degree measure of an angle from a given radian measure, multiply (see Sec. 180) the number of radians by $\frac{180^\circ}{\pi} \approx 57^\circ.296$, that is, by 57°17′45″ (if the required accuracy is 0.5′ and the angle contains no more than 2 radians, the multiplier may be rounded off to 57°.30 since the error of 0.004 degree constitutes about one fourth of a minute).

**Example 3.** Find the degree measure of an angle containing 1.360 radians (to within 1′).

**Solution.** $1.360 \cdot 57^\circ.30 = 77^\circ.93 = 77^\circ 56'$.

The computations can be simplified by using the table of Sec. 9, page 49. We find

$$\begin{array}{lll} 1 & \text{radian} = & 57^\circ 18' \\ 0.3 & \text{radian} = & 17^\circ 11' \\ 0.060 & \text{radian} = & 3^\circ 26' \\ & & \overline{77^\circ 55'} \end{array}$$

The discrepancy of 1′ is due to an accumulation of errors of the terms (see Sec. 52).

**Example 4.** Find the degree measure of an angle containing 6.485 radians. Using the table we find

$$\begin{array}{lll} 6 & \text{radians} = & 343^\circ 46' \\ 0.4 & \text{radian} = & 22^\circ 55' \\ 0.08 & \text{radian} = & 4^\circ 35' \\ 0.005 & \text{radian} = & 0^\circ 17' \\ & & \overline{371^\circ 33'} \end{array}$$ (limiting error, 2′)

### 182. Trigonometric Functions of an Acute Angle

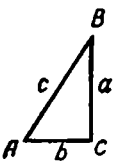

Fig. 219.

The solution of any triangle ultimately reduces to the solution of right triangles. In a right triangle $ABC$, the ratio of two sides, say side $a$ to the hypotenuse $c$, is wholly dependent on the value of one of the acute angles, say $A$ (Fig. 219). The ratios of different pairs of sides of a right triangle are called *trigonometric functions* of its acute angle. With respect to angle $A$, these functions have the following names and designations:

(1) sine: $\sin A = \frac{a}{c}$ (ratio of opposite side to hypotenuse)

(2) cosine: $\cos A = \frac{b}{c}$ (ratio of adjacent side to hypotenuse),

(3) tangent: $\tan A = \frac{a}{b}$ (ratio of opposite side to adjacent side),

(4) cotangent: $\cot A = \frac{b}{a}$ (ratio of adjacent side to opposite side),

(5) secant: $\sec A = \frac{c}{b}$ (ratio of hypotenuse to adjacent side),

(6) cosecant: $\csc A = \frac{c}{a}$ (ratio of hypotenuse to opposite side).

With respect to the angle $B$ (the complementary angle with respect to $A$) the names are appropriately changed as follows:

$$\sin B = \frac{b}{c}, \quad \cos B = \frac{a}{c}, \quad \tan B = \frac{b}{a},$$

$$\cot B = \frac{a}{b}, \quad \sec B = \frac{c}{a}, \quad \csc B = \frac{c}{b}$$

For certain angles we can write the exact expressions of their trigonometric functions. The most important cases are given in the table below. *

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* Strictly speaking, angles 0° and 90° cannot appear in a right triangle as acute angles, but in an extended view of trigonometric functions (see below) we consider the values of the trigonometric functions of these angles as well. On the other hand, one of the acute angles of a triangle can approach 90° as close as desired, in which case the other angle will approach zero; then the corresponding trigonometric quantities will approach the values indicated in the table.

The symbol $\infty$ which we find in the tables indicates that the absolute value of the given quantity increases without bound when the angle approaches the value indicated in the table. This is what is meant when we say that a quantity is equal to infinity, or becomes infinite (cf. Sec. 37 and Sec. 217).

| $A$ | $\sin A$ | $\cos A$ | $\tan A$ | $\cot A$ | $\sec A$ | $\csc A$ |
|---|---|---|---|---|---|---|
| 0° | 0 | 1 | 0 | $\infty$ | 1 | $\infty$ |
| 30° | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ | $\sqrt{3}$ | $\frac{2}{\sqrt{3}}$ | 2 |
| 45° | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | 1 | $\sqrt{2}$ | $\sqrt{2}$ |
| 60° | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{1}{\sqrt{3}}$ | 2 | $\frac{2}{\sqrt{3}}$ |
| 90° | 1 | 0 | $\infty$ | 0 | $\infty$ | 1 |

This table is of more theoretical than practical value since it contains roots that cannot be extracted exactly. For most angles, it is impossible to write the exact numerical values of the trigonometric functions even with the aid of roots. However, approximate values can be computed to within any desired degree of accuracy (see Sec. 203). The computations are arduous and so have been done once and for all time and the results have been tabulated. Four-place tables of sines and cosines are given on pages 36-39 (Sec. 6), tables of tangents and cotangents on pages 40-47 (Sec. 7).

### 183. Finding a Trigonometric Function from an Angle *

**(a) Sine and cosine.** In the table of Sec. 6 (pages 36-39) are given the sines of all angles from 0° to 90° at 1′ intervals to four decimal places. Since the sine of an angle equals the cosine of the complementary angle (Sec. 182), the same table may be used to find the cosines of all angles from 90° to 0° for 1′ intervals.

* If radian measure of the angle is given, convert to degree measure (see Sec. 181).

When looking for the sine, the number of degrees is read in the *left-hand* column "Degrees" and the rounded number of minutes (0′, 10′, 20′, 30′, 40′, 50′) *at the top* (this is indicated by the heading "sines" above the table). When seeking the cosine, the number of degrees is read in the *right-hand* column "Degrees" and the rounded number of minutes *at the bottom* (this is indicated by the footing "cosines"). We obtain the basic result at the intersection of the proper row and column. Interpolate for the missing number of minutes (from 1 to 9). This is done in the section "Proportional Parts" in the row where the basic result is obtained. If the sine is being sought, the interpolation *is added* to the basic result; if the cosine is being sought, the interpolation *is subtracted* from the basic result (this is because the sine increases as the angle increases and the cosine decreases).

**Example 1.** Find $\sin 53°40'$.

In the *left-hand* column "Degrees" take the number 53 and 40′ in the *top* row. At their intersection we find 0.8056. No proportional parts data are needed:

$$\sin 53°40' = 0.8056$$

**Example 2.** Find $\cos 63°10'$.

Take the number 63 in the *right-hand* column "Degrees" and 10′ in the *bottom* row. Their intersection yields 0.4514. No proportional parts data are needed:

$$\cos 63°10' = 0.4514$$

**Example 3.** Find $\sin 62°24'$.

Take 62 in the *left-hand* column and 20′ in the *top* row. The intersection yields the basic result of 0.8857. Move along the same row to the proportional parts data (column 4′) to find 5 (which is 0.0005). Adding it to the basic result we get 0.8862.

**Work:**

$$\begin{array}{r} \sin 62°20' = 0.8857 \\ +4' \qquad +5 \\ \hline \sin 62°24' = 0.8862 \end{array}$$

**Example 4.** Find $\cos 42°16'$.

Take 42 in the *right-hand* column and 10′ in the *bottom* row. The intersection of column and row yields the basic result of 0.7412. In the same row, move to the proportional parts data (column 6′) to get the number 12. Subtracting it from the basic result we obtain 0.7400.

**Work:**

$$\begin{array}{r} \cos 42°10' = 0.7412 \\ +6' \quad -12 \\ \hline \cos 42°16' = 0.7400 \end{array}$$

**(b) Tangent and cotangent.** In the table of Sec. 7 (pages 40-47) are given the tangents of all angles from 0° to 90° at 1′ intervals accurate to the fourth significant digit. Between 0° and 76° the table is constructed like the table of sines. In the interval between 76° and 90° (where the variation of the tangent is extremely non-uniform) there is no proportional data section, but the table is given in more detail.

Since the tangent of an angle equals the cotangent of the complementary angle (Sec. 182), the same table may be used to find the cotangents of all angles from 90° to 0° at 1′ intervals. In looking up a tangent, the value of the angle is read like that of sines in Sec. 6 [see Item (a)]; the cotangent is sought in the same way as the cosine.

**Example 1.** Find tan 82°18′.

Read angles ("Degrees") in the *left* column, 82°10′, and 8′ in the *top* row. The intersection yields

$$\tan 82°18' = 7.396$$

**Example 2.** Find cot 12°35′.

Read up on the *right*, 12°30′, and take 5′ in the *bottom* row. The intersection of column and row yields

$$\cot 12°35' = 4.480$$

**Example 3.** Find cot 58°36′.

Reading up on the right we have 58°, bottom row, 30′. The intersection yields 0.6128. In the same row, move to the proportional parts data (column 6′ at the bottom) to find 24. Subtract it from 0.6128 to get 0.6104.

**Work:**

$$\begin{array}{r} \cot 58°30' = 0.6128 \\ +6' \quad -24 \\ \hline \cot 58°36' = 0.6104 \end{array}$$

**Example 4.** Find tan 48°43′.

**We find**

$$\begin{array}{r} \tan 48°40' = 1.1369 \\ +3' \quad +20 \\ \hline \tan 48°43' = 1.1389 \end{array}$$

### 184. Finding an Angle from a Trigonometric Function

To find an angle from a given sine or cosine, use the tables of Sec. 6 (pages 36-39); from a given tangent or cotangent, the tables of Sec. 7 (pages 40-47). Run down one of the columns (say the column headed 0′ at the top), and find the value we desire or the nearest value. In the former case, we read off the value of the required angle directly (using the left column of degrees and the top row of minutes when dealing with sine or tangent, and the right column and bottom row when dealing with cosine or cotangent; cf previous section). In the latter case, we check for a closer value in the vicinity; if there is one, we read off the value of the angle as above; if there is none, we take the value found. If necessary we refer to proportional parts data. Bear in mind that the proportional parts figure is positive for increasing sine and tangent, and negative for increasing cosine and cotangent (if required, degree measure may be changed to radian measure; see Sec. 181).

**Example 1.** What is the acute angle $\alpha$ whose $\cos = 0.7173$?

In the table of Sec. 6, run down the column headed 0′ to find the value 0.7193, which is close to the given value. Near it we find 0.7173 which coincides with the given value. Read the degrees in the right column and the minutes in the bottom row to find $\alpha = 44°10'$.

**Example 2.** Find the acute angle $\alpha$ whose $\cos = 0.2643$.

In the table of Sec. 6, the nearest value is 0.2644. The difference is 0.0001, and the proportional parts section has the smallest number 3 (corresponding to 1′). Hence we disregard this correction. Using the right column for degrees and the bottom row for minutes, we get $\alpha = 74°40'$.

**Example 3.** Find the acute angle $\alpha$, if $\cos\alpha = 0.7458$.

The nearest tabulated value is 0.7451, corresponding to an angle of 41°50′. The given value exceeds the tabulated value by 7 units of the fourth decimal place. In the same row, moving to the proportional parts data, we find 6 and 8. Take either one and *subtract* from the angle 41°50′ the correction 3′ or 4′. This yields 41°47′ (too large) or 41°46′ (too small).

**Work:**

$$\begin{array}{r} 0.7451 = \cos 41°50' \\ +7 \qquad\quad -3' \\ \hline 0.7458 = \cos 41°47' \end{array}$$

**Example 4.** What is the acute angle $\alpha$ whose tangent is 4.827?

In the table of Sec. 7, find a nearest value (too small), 4.822 and another nearest value 4.829 (too large). Since the latter is closer to the given one than the former, take the latter. In the left column we read 78°10′, in the top row, 8′. This yields $\alpha = 78°18'$.

### 185. Solving Right Triangles

**1. By two sides.** If two sides of a right triangle are given, the third side can be found by the Pythagorean theorem (Sec. 147). The acute angles are found from one of the first three formulas of Sec. 182 depending on which sides are given. If, say, we are given the legs $a$, $b$, then the acute angle $A$ is found from the formula

$$\tan A = \frac{a}{b}$$

and the acute angle $B$ is found from the formula $B = 90° - A$.

**Case 1.** Given leg $a = 0.528$ metre and hypotenuse $c = 0.697$ metre.

(1) Determine leg $b$:

$$b = \sqrt{c^2 - a^2} = \sqrt{0.697^2 - 0.528^2} \approx 0.455 \text{ (m)}$$

(2) Determine angle $A$:

$$\sin A = \frac{a}{c} = \frac{0.528}{0.697} \approx 0.757$$

In the table of sines find $A \approx 49°10'$ (limiting error of 5′). It is meaningless to find $A$ to within one minute since by regarding the values of $a$ and $c$ as approximate numbers we cannot be sure even of the third digit of the quotient $\frac{a}{c} \approx 0.757$ (Sec. 56).

(3) Determine angle $B$:

$$B = 90° - A \approx 90° - 49°10' = 40°50'$$

**Case 2.** Given legs $a = 8.3$ cm, $b = 12.4$ cm.

(1) Determine hypotenuse $c$:

$$c = \sqrt{a^2 + b^2} = \sqrt{8.3^2 + 12.4^2} \approx 14.9 \text{ (cm)}$$

(2) Determine angle $A$:

$$\tan A = \frac{a}{b} = \frac{8.3}{12.4} \approx 0.67, \quad A \approx 34°$$

(3) Determine angle $B$:

$$B = 90° - A \approx 90° - 34° = 56°$$

**2. By a side and an acute angle.** If we are given an acute angle $A$, then $B$ is found by the formula $B = 90° - A$. The sides may be found from the formulas of Sec. 182, which may be represented as:

$$a = c \sin A, \quad b = c \cos A, \quad a = b \tan A,$$
$$b = c \sin B, \quad a = c \cos B, \quad b = a \tan B$$

Choose the formulas that contain the given side or the side found.

**Case 3.** Given the hypotenuse $c = 79.79$ metres and the acute angle $A = 66°36'$.

(1) Determine angle $B$: $B = 90° - A = 90° - 66°36' = 23°24'$.

(2) Determine leg $a$: $a = c \sin A = 79.79 \cdot \sin 66°36' = 79.79 \cdot 0.9178 \approx 73.23$ (m).

(3) Determine leg $b$: $b = c \cos A = 79.79 \cdot 0.3971 \approx 31.68$ ($m$).

**Case 4.** Given leg $a = 12.3$ metres and acute angle $A = 63°00'$.

(1) Determine angle $B$: $B = 90° - 63°00' = 27°00'$.

(2) Determine leg $b$: $b = a \tan B = 12.3 \tan 27°00' = 12.3 \cdot 0.509 \approx 6.26$ (m).

(3) Determine hypotenuse $c$: $c = \frac{a}{\sin A} = \frac{12.3}{\sin 63°00'} = \frac{12.3}{0.891} \approx 13.8$ (m).

### 186. Table of Logarithms of Trigonometric Functions

Solving right triangles always requires multiplication and division. If considerable accuracy is required (say, four-digit numbers are being multiplied), then the operations are very time-consuming, arduous and, hence, the possibility of errors cropping up increases. The use of logarithms saves both time and energy. In logarithmic computations, we do not use tables of trigonometric functions but tables of their logarithms. This results in a great saving of time since

instead of looking up the sine of an angle in the table of trigonometric functions and then finding the logarithm of the sine in the table of logarithms, we find the logarithm of the sine directly.

In the table of Sec. 5 (pages 14-17) are given the values of the logarithms of sine, cosine, tangent and cotangent to four-decimal-place accuracy for 10′ intervals. If the angle does not exceed 45°, the name of the required function is read at the top and the angle at the left. If the angle exceeds 45°, the name of the function is read at the bottom, and the value of the angle on the right.

The same table permits computing logarithms of trigonometric functions for 1′ intervals as well. The mode of computation (see Secs. 187 and 188) is based on the fact that, within the range of 10′, the variation of angle is proportional to the variation of log sin, log tan, log cos, and log cot. As a rule, the error due to this assumption does not affect the fourth decimal place. The only exceptions are log sin and log tan for angles close to 0° (from 0° to 4°) and log cos and log cot for angles close to 90° (86° to 90°); in these cases the error becomes perceptible.

Let us take an illustrative example. The increase in an angle from 12°20′ to 12°30′ is associated with an increase in log sin from $\bar{1}.3296$ to $\bar{1}.3353$, which is 0.0057. An increase twice this,* from 12°20′ to 12°40′ is associated with an increase of log sin from $\bar{1}.3296$ to $\bar{1}.3410$, which is 0.0114. This increase is double the previous increase.

Changes in log sin that correspond to an increase in angle of 10′ need not be computed since they are given in the columns headed d (which stands for difference). Thus, in column log sin opposite 12°20′ we read $\bar{1}.3296$, opposite 12°30′, $\bar{1}.3353$. The difference $\bar{1}.3353-\bar{1}.3296=0.0057$ is given in the left column d between $\bar{1}.3296$ and $\bar{1}.3353$ (for brevity, only 57 is given).

The same differences (with the minus sign this time, however) yield variations of log cos corresponding to 10′ increases in angle. Thus, the same 57 gives a *decrease* in log cos as the angle increases from 77°30′ to 77°40′.

For log tan and log cot, the differences are given in the middle column headed c d (common difference). They serve the two adjacent columns on the right and left. For example, the differences log tan 12°30′ — log tan 12°20′ and log

* We took an increase in excess of 10′ so as not to have to resort to a more detailed table.

tan 77°40′ — log tan 77°30′ have a common value 0.0061, which is given in the column c d between the appropriate rows. The number 0.0061 also gives a *decrease* in log cot as the angle increases from 12°20′ to 12°30′ and from 77°30′ to 77°40′.

The numbers given in the columns d and c d are called *tabular differences*.

### 187. Finding the Logarithm of a Trigonometric Function from the Angle*

For angles with a round number of minutes (0′, 10′, 20′, 30′, 40′, 50′), the required quantity (to within 0.0001) is taken directly from the table of Sec. 5 described in the previous section. For the other angles, interpolation is required.

Here, remember that for sin and tan the signs of the angle corrections and the logarithms of the trigonometric function are the same, while for cos and cot they are different.

**Example 1.** Find log cos 24°13′.

The angle is less than 45° and so enter the table at the *top*, "log cos". There** we find log cos 24°10′ = $\bar{1}$.9602. The tabular difference (the number in the right column d) is log cos 24°10′ — log cos 24°20′ = 0.0006. Let us find the correction $x$ for 3′. From the proportion

$$x = 0.0006 = 3' : 10'$$

we have

$$x = 0.0006 \cdot 0.3 \approx 0.0002$$

This correction must *be subtracted* from $\bar{1}$.9602. We get

$$\log \cos 24°13' = \bar{1}.9600$$

**Work:**

$$\begin{array}{l} \log \cos 24°10' = \bar{1}.9602 \quad d = 6 \\ \qquad\qquad +3' \qquad -2 \\ \hline \log \cos 24°13' = \bar{1}.9600 \end{array}$$

---

* If the angle is given in radian measure, convert to degree measure first (Sec. 181).

** Remember that in the table of Sec. 5 the characteristics of all logarithms are increased by 10, therefore, in place of $\bar{1}$ we find 9, in place of $\bar{2}$, we find 8, etc.

**Note.** When interpolating, it is not necessary to write out the whole procedure. It is enough to multiply (mentally) the number of minutes by the tabular difference and, rounding off the product, to drop the zero at the end. In our case, we have to multiply 3 by 6 and round the product 18 to 20. Dropping the zero, we have the correction: 2.

**Example 2.** Find log tan 57°48′.

The given angle exceeds 45° and so we enter the table *at the bottom* in the column headed log tan and take log tan 57°50′ = 0.2014; c d (= log tan 57°50′ − log tan 57°40′) = 28 (i.e., 0.0028). We now need the correction for the *lacking* 2′. Multiply (see note of Example 1) 2 by 28 to get 60 (approximately). Drop the zero and get the correction 6. Subtract it from 0.2014 to get log tan 57°48′ = 0.2008.

**Work:**

$$\begin{array}{rl} \log \tan 57°50' = 0.2014 & d = 28 \\ \quad -2' \cdot \quad -6 & \\ \hline \log \tan 57°48' = 0.2008 & \end{array}$$

**Note.** It is also possible to take log tan 57°40′ = 0.1986 from the table, find the correction 22 (8.28 ≈ 220) for 8, and add it to 0.1986. The result is the same but it is easier to multiply 28 by 2 than by 8 so there is less chance of mistakes when multiplying mentally.

**Example 3.** Find log cot 65°17′.

$$\begin{array}{rl} \log \cot 65°20' = \bar{1}.6620 & d = 34 \\ \quad -3' \quad +10 & \\ \hline \log \cot 65°17' = \bar{1}.6630 & \end{array}$$

**Example 4.** Find log sin 40°34′.

$$\begin{array}{rl} \log \sin 40°30' = \bar{1}.8125 & d = 15 \\ \quad +4' \quad +6 & \\ \hline \log \sin 40°34' = \bar{1}.8131 & \end{array}$$

### 188. Finding the Angle from the Logarithm of the Trigonometric Function

Run down the appropriate columns of the table in Sec. 5 (the values of each function are given in two columns) and find the required value or the nearest value; in the latter case, write out the tabular difference. If the name

of the trigonometric function is given at the top, read degrees and tens of minutes on the left; if the name is given at the bottom, read them on the right. Finally, interpolate if necessary by means of a proportional calculation (the angle correction is of the same sign for sin and tan, and different sign for cos and cot).

**Example 1.** Find the acute angle $\alpha$ if $\log \tan \alpha = 0.2541$. The value 0.2533, which is nearest to the given one (tabular difference $cd = 29$), lies in the column log tan reading up. We therefore read $60°50'$ *on the right*. The correction $x$ for the extra 8 units of the last place ($0.2541 - 0.2533 = 0.0008$) is found from the proportion

$$x : 10' = 8 : 29$$

whence $x = \frac{10' \cdot 8}{29} \approx 3'$. Adding this correction, we get $\alpha = 60°53'$.

**Work:**

$$\begin{array}{rl} \log \tan \alpha = 0.2541 & \\ 0.2533 = \log \tan\ 60°50' & d = 29 \\ +8 \qquad\qquad +3' & \\ \hline 0.2541 = \log \tan\ 60°53' & \\ \alpha = 60°53' & \end{array}$$

**Note.** The correction may be found mentally in the following manner. Consider the difference between the given value and the tabular value—in our case, 0.0008—as a whole number 8 (that is, disregard decimal point and zeros on the left). Increase it tenfold (80) and divide by the tabular difference (29). The quotient (rounded to units)—in our case it is 3—yields the correction in minutes.

**Example 2.** Evaluate the acute angle $\alpha$ if $\log \cos \alpha$ is $\bar{1}.4361$.

The nearest tabulated value is $\bar{1}.4359$; the tabular difference $d = 44$. The heading log cos is at the bottom, and so we read $74°10'$ *on the right*. The tenfold difference between the given value and the tabulated value is 20. The quotient of $\frac{20}{44}$ (less than half) is rounded off to zero.

Hence $\alpha = 74°10'$.

**Example 3.** Evaluate the acute angle $\alpha$ for $\log \cot \alpha = \bar{1}.6780$.

The nearest tabulated value is $\bar{1}.6785$; the tabular difference is 32. The heading log cot is at the bottom, and so we read $64°30'$ *on the right*. The given value is less by 5 than the tabulated value. Divide the tenfold increased number,

50, by 32. The rounded off quotient is 2. Add 2′ to get $\alpha = 64°32'$.

**Work:**

$$\log \cot \alpha = \bar{1}.6780$$
$$\bar{1}.6785 = \log \cot 64°30' \quad d = 32$$
$$-5 \qquad +2'$$
$$\bar{1}.6780 = \log \cot 64°32'$$
$$\alpha = 64°32'$$

**Example 4.** Find the acute angle $\alpha$ if $\log \sin \alpha = \bar{1}.7414$.

$$\log \sin \alpha = \bar{1}.7414$$
$$\bar{1}.7419 = \log \sin 33°30' \quad d = 19$$
$$-5 \qquad -3'$$
$$\bar{1}.7414 = \log \sin 33°27'$$
$$\alpha = 33°27'$$

### 189. Solving Right Triangles by Logarithms

**Case 1.** Given: hypotenuse $c = 9.994$, leg $b = 5.752$. Determine $a$, $B$, $A$.

(1) Determine $B$: $\sin B = \frac{b}{c}$,

$$\log b = 0.7598$$
$$-\log c = \bar{1}.0003$$
$$\log \sin B = \bar{1}.7601; \quad B = 35°8'$$

(2) Determine $A$: $A = 90° - B = 54°52'$.

(3) Determine $a$: $a = b \tan A$,

$$\log b = 0.7598$$
$$\log \tan A = 0.1526$$
$$\log a = 0.9124, \quad a = 8.173$$

**Case 2.** Given legs $a = 0.920$ and $b = 0.849$. Determine the hypotenuse and acute angles.

(1) Determine angle $B$: $\tan B = \frac{b}{a}$,

$$\log b = \bar{1}.9289$$
$$-\log a = 0.0362$$
$$\log \tan B = \bar{1}.9651, \quad B = 42°42'$$

(2) Determine angle $A$: $A = 90° - B = 47°18'$.

(3) Determine the hypotenuse $c$: $c=\frac{b}{\sin B}$,

$$\begin{aligned}\log b &= \bar{1}.9289\\ -\log\sin B &= 0.1687\\ \hline \log c &= 0.0976,\quad c=1.252\end{aligned}$$

**Case 3.** Given: hypotenuse $c=798.1$, acute angle $A=49°18'$. Determine $a$, $b$, $B$.

(1) Determine $B$: $B=90°-49°18'=40°42'$.

(2) Determine $a$: $a=c\sin A$,

$$\begin{aligned}\log c &= 2.9021\\ \log\sin A &= \bar{1}.8797\\ \hline \log a &= 2.7818,\quad a=605.1\end{aligned}$$

(3) Determine $b$: $b=c\sin B$,

$$\begin{aligned}\log c &= 2.9021\\ \log\sin B &= \bar{1}.8143\\ \hline \log b &= 2.7164,\quad b=520.5\end{aligned}$$

**Case 4.** Given leg $a=324.6$; acute angle $B=49°28'$. Determine $b$, $c$, $A$.

(1) Determine $A$:

$$A=90°-B=90°-49°28'=40°32'$$

(2) Determine $b$: $b=a\tan B$,

$$\begin{aligned}\log a &= 2.5113\\ \log\tan b &= 0.0680\\ \hline \log b &= 2.5793,\quad b=379.6\end{aligned}$$

(3) Determine $c$: $c=\frac{a}{\sin A}$,

$$\begin{aligned}\log a &= 2.5113\\ -\log\sin A &= 0.1872\\ \hline \log c &= 2.6985,\quad c=499.5\end{aligned}$$

### 190. Practical Uses of Right-Triangle Solutions

In order to make effective use of the procedures discussed above, it is necessary to learn to use the relevant tables and accurately find the needed results. But this is not all: there are two other difficulties. The first is of a purely geometrical nature; to learn to find a simple method of isolating a right-angle triangle in any given geometric figure. The following illustrative examples will help.

**Example 1.** Referring to the isosceles triangle $ABC$ in Fig. 220, the base $AC$ and side $AB$ are known. It is required to determine angle $B$ at the vertex.

Draw the altitude $BD$ which bisects base $AC$ and angle $B$. Knowing $AC$, we find $AD=\frac{AC}{2}$. In the right triangle $ABD$, we find $\angle ABD$ from side $AD$ and hypotenuse $AB$ (Case 1, Secs. 185 and 189). Multiplying by two, we find the desired vertex angle.

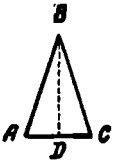


Fig. 220.

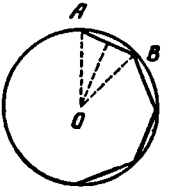


Fig. 221.

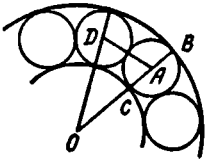


Fig. 222.

**Example 2.** Given the radius $R$ of a circle, to compute the side $AB$ of a regular inscribed nonagon.

In Fig. 221, draw radii $OA$, $OB$ to the extremities of the chord $AB$ to get an isosceles triangle in which the side $OA=R$ is known. It is also easy to find the vertex angle $AOB=\frac{360°}{9}=40°$. Dividing $\angle AOB$ into two right triangles by drawing the altitude as we did in the preceding problem, reduce the problem to Case 3 of Secs. 185 and 189.

Another difficulty—the most essential one—is to *translate a specifically stated problem into mathematical language.*

**Example 3.** Compute the inner and outer radii of a ball bearing so that it can accommodate twenty steel balls of diameter 16 mm each.

(To simplify the problem we assume that the balls are packed tightly).

The main difficulty here is to isolate the mathematical content. Constructing Fig. 222, we note that we know the diameter of a ball, $BC=16$ mm, and hence its radius $AB=AC=8$ mm. Besides, the angle between the radii $OA$ and $OD$ from the centres of adjacent balls must be $\frac{360°}{20}=18°$. Furthermore, the line $AD$ which connects the centres of adja cent balls must be equal to the diameter of each of them, which is $AD=16$ mm. Now we have an isosceles triangle,

$AOD$, in which we know the base $AD=16$ mm and the vertex angle $AOD=18°$. Dividing it into two right triangles, we reduce the problem to Case 4, Sec. 189, and obtain $OD=OA=51.1$ mm. Whence we find the outer radius:

$$OB=OA+AB=51.1+8=59.1 \text{ mm}$$

and the inner radius:

$$OC=OA-AC=43.1 \text{ mm}$$

### 191. Fundamental Relations of Trigonometry

Knowing one of the trigonometric functions of an acute angle, it is possible, by applying the relations (identities) given below, to determine the others. However, their main value lies in the possibility of substantially simplifying the aspect of many general formulas and thus reducing the computational process.

$$\sin^2\alpha+\cos^2\alpha=1, \quad \tan\alpha\cdot\cot\alpha=1,$$

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}, \qquad \cot\alpha=\frac{\cos\alpha}{\sin\alpha},$$

$$\sin\alpha\cdot\csc\alpha=1, \qquad \cos\alpha\cdot\sec\alpha=1,$$

$$\sec^2\alpha=1+\tan^2\alpha, \qquad \csc^2\alpha=1+\cot^2\alpha,$$

$$\cos^2\alpha=\frac{1}{1+\tan^2\alpha}=\frac{\cot^2\alpha}{1+\cot^2\alpha},$$

$$\sin^2\alpha=\frac{1}{1+\cot^2\alpha}=\frac{\tan^2\alpha}{1+\tan^2\alpha}$$

These formulas hold true for the trigonometric functions of any angle (see next section). They are called trigonometric identities.

### 192. Trigonometric Functions of an Arbitrary Angle

It is possible to construct the whole of trigonometry using only the trigonometric functions of acute angles. But then in the solution of oblique triangles and in many other problems requiring trigonometry we would have to distinguish a multitude of separate cases of one and the same problem, depending on the magnitude of the given angle. In contrast, the solution of all problems becomes unified if we extend, as follows, the concept of sine, cosine, etc. to angles of arbitrary size, that is, not only between 0° and 180° but

exceeding 180°, not only positive angles, but negative angles as well (see Sec. 142).

To reckon angles, we take a circle $ABA'B'$ (Fig. 223) with two mutually perpendicular diameters $AA'$ (first) and $BB'$ (second). Arcs will be reckoned from point $A$. Counterclockwise rotation will be taken as the positive direction.

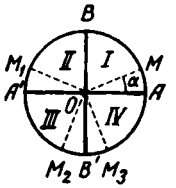


Fig. 223.

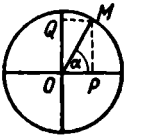


Fig. 224.

An angle $\alpha$ is formed between the moving radius $OM$ and the fixed radius $OA$. This angle can lie in the first quadrant ($MOA$), the second ($M_1OA$), the third ($M_2OA$) or the fourth ($M_3OA$). Taking as positive the directions $OA$, $OB$ and as negative the directions $OA'$, $OB'$, we define the trigonometric functions of the angles as follows:

Fig. 225

Fig. 226.

The *line of sine* of angle $\alpha$ (Fig. 224) is the projection $OQ$ of the moving radius on the second diameter (taken with appropriate sign).

The *line of the cosine* $OP$ is the projection of the moving radius on the first diameter.

The *sine* of angle $\alpha$ (Fig. 224) is the ratio of the line of the sine $OQ$ (taken with appropriate sign) to the radius $R$ of the circle.

The *cosine* is the ratio of the line of the cosine $OP$ (taken with appropriate sign) to the radius.

Figure 225 gives the signs of *the sine of* angle $\alpha$ (Fig. 226, the signs of the cosine) in different quadrants.

## Expressing One Trigonometric Function in Terms of Another

| | sin | cos | tan | cot | sec | csc |
|---|---|---|---|---|---|---|
| sin $x$ | | $= \pm\sqrt{1-\cos^2 x}$ | $= \dfrac{\tan x}{\pm\sqrt{1+\tan^2 x}}$ | $= \dfrac{1}{\pm\sqrt{1+\cot^2 x}}$ | $= \dfrac{\pm\sqrt{\sec^2 x-1}}{\sec x}$ | $= \dfrac{1}{\csc x}$ |
| cos $x$ | $= \pm\sqrt{1-\sin^2 x}$ | | $= \dfrac{1}{\pm\sqrt{1+\tan^2 x}}$ | $= \dfrac{\cot x}{\pm\sqrt{1+\cot^2 x}}$ | $= \dfrac{1}{\sec x}$ | $= \dfrac{\pm\sqrt{\csc^2 x-1}}{\csc x}$ |
| tan $x$ | $\dfrac{\sin x}{\pm\sqrt{1-\sin^2 x}}$ | $= \dfrac{\pm\sqrt{1-\cos^2 x}}{\cos x}$ | | $= \dfrac{1}{\cot x}$ | $= \pm\sqrt{\sec^2 x-1}$ | $= \dfrac{1}{\pm\sqrt{\csc^2 x-1}}$ |
| cot $x$ | $= \dfrac{\pm\sqrt{1-\sin^2 x}}{\sin x}$ | $= \dfrac{\cos x}{\pm\sqrt{1-\cos^2 x}}$ | $= \dfrac{1}{\tan x}$ | | $= \dfrac{1}{\pm\sqrt{\sec^2 x-1}}$ | $= \pm\sqrt{\csc^2 x-1}$ |
| sec $x$ | $= \dfrac{1}{\pm\sqrt{1-\sin^2 x}}$ | $= \dfrac{1}{\cos x}$ | $= \pm\sqrt{1+\tan^2 x}$ | $= \dfrac{\pm\sqrt{1+\cot^2 x}}{\cot x}$ | | $= \dfrac{\csc x}{\pm\sqrt{\csc^2 x-1}}$ |
| csc $x$ | $= \dfrac{1}{\sin x}$ | $= \dfrac{1}{\pm\sqrt{1-\cos^2 x}}$ | $= \dfrac{\pm\sqrt{1+\tan^2 x}}{\tan x}$ | $= \pm\sqrt{1+\cot^2 x}$ | $= \dfrac{\sec x}{\pm\sqrt{\sec^2 x-1}}$ | |

The *line of the tangent* ($AD_1$, $AD_2$, etc., Fig. 227) is a segment of the tangent line drawn through the extremity $A$ of the first diameter from the point of tangency to intersection with the moving radius ($OM_1$, $OM_2$, etc.) produced.

The *line of the cotangent* ($BE_1$, $BE_2$, etc., Fig. 228) is a segment of the tangent line drawn through the extremity $B$ of the second diameter from the point $B$ of tangency to intersection with the moving radius ($OM_1$, $OM_2$, etc.) produced.

The *tangent* of an angle is the ratio of the line of the tangent (taken with appropriate sign) to the radius.

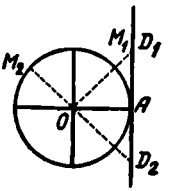


Fig. 227.

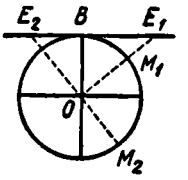


Fig. 228.

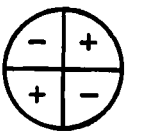


Fig. 229.

The *cotangent* is the ratio of the line of the cotangent (taken with appropriate sign) to the radius.

The signs of tangent and cotangent for the various quadrants are indicated in Fig. 229.

It is simplest to define the secant and cosecant as reciprocals of the cosine and sine.

The table on page 351 gives the expressions of each trigonometric function of any angle in terms of the other functions. In the expressions with two signs, the choice of sign depends on the quadrant in which the angle lies (see Figs. 225, 226, 229).

The graphs of trigonometric functions are given in Sec. 213.

### 193. Reduction Formulas

These are formulas which make it possible (1) to find numerical values of trigonometric functions of angles exceeding 90°, and (2) to make transformations which simplify the aspect of formulas.

All formulas hold true for arbitrary angles $\alpha$, although they are mainly used when $\alpha$ is an acute angle.

**First Group:**

$$\sin(-\alpha) = -\sin\alpha, \quad \tan(-\alpha) = -\tan\alpha,$$
$$\cot(-\alpha) = -\cot\alpha, \quad \cos(-\alpha) = +\cos\alpha$$

These formulas permit dispensing with negative angles.

**Second Group:**

$$\left.\begin{matrix}\sin\\ \cos\\ \tan\\ \cot\end{matrix}\right\} (360°k+\alpha) = \left.\begin{matrix}\sin\\ \cos\\ \tan\\ \cot\end{matrix}\right\} \alpha \quad (k \text{ positive integer})$$

These formulas enable us to avoid considering angles greater than 360°.

**Third Group:**

$$\left.\begin{matrix}\sin\\ \cos\\ \tan\\ \cot\end{matrix}\right\} (180° \pm \alpha) = \left.\begin{matrix}\mp\sin\\ -\cos\\ \pm\tan\\ \pm\cot\end{matrix}\right\} \alpha$$

The names of the functions are preserved; the sign on the right is that which the left-hand side has for $\alpha$ an acute angle.

For example, $\sin(180°-\alpha) = +\sin\alpha$, since for $\alpha$ acute, $180°-\alpha$ lies in the second quadrant where the sine is positive; $\sin(180°+\alpha) = -\sin\alpha$, since for $\alpha$ acute, the angle $180°+\alpha$ lies in the third quadrant where the sine is negative; $\cos(180°-\alpha) = -\cos\alpha$, since the cosine in the second quadrant is negative, etc.

**Fourth Group:**

$$\left.\begin{matrix}\sin\\ \cos\\ \tan\\ \cot\end{matrix}\right\} (90° \pm \alpha) = \left.\begin{matrix}+\cos\\ \mp\sin\\ \mp\cot\\ \mp\tan\end{matrix}\right\} \alpha, \quad \left.\begin{matrix}\sin\\ \cos\\ \tan\\ \cot\end{matrix}\right\} (270° \pm \alpha = \left.\begin{matrix}-\cos\\ \pm\sin\\ \mp\cot\\ \mp\tan\end{matrix}\right\} \alpha$$

The name of the function varies: the cofunction is taken instead of the function. The rule of signs is the same as in the preceding group. For instance, $\cos(270°-\alpha) = -\sin\alpha$ since the angle $270°-\alpha$ for $\alpha$ acute belongs to the third quadrant where the cosine is negative; $\cos(270°+\alpha) = +\sin\alpha$ since the cosine is positive in the fourth quadrant.

All these formulas may be obtained by applying the following rule.

Any trigonometric function of an angle $90°n+\alpha$ is equal in absolute value to the same function of the angle if $n$ is even and to the cofunction if $n$ is odd. If the function of

| Functions | Angles | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $-\alpha$ | $90° - \alpha$ | $90° + \alpha$ | $180° - \alpha$ | $180° + \alpha$ | $270° - \alpha$ | $270° + \alpha$ | $360°k - \alpha$ | $360°k + \alpha$ |
| sin | $-\sin \alpha$ | $+\cos \alpha$ | $+\cos \alpha$ | $+\sin \alpha$ | $-\sin \alpha$ | $-\cos \alpha$ | $-\cos \alpha$ | $-\sin \alpha$ | $+\sin \alpha$ |
| cos | $+\cos \alpha$ | $+\sin \alpha$ | $-\sin \alpha$ | $-\cos \alpha$ | $-\cos \alpha$ | $-\sin \alpha$ | $+\sin \alpha$ | $+\cos \alpha$ | $+\cos \alpha$ |
| tan | $-\tan \alpha$ | $+\cot \alpha$ | $-\cot \alpha$ | $-\tan \alpha$ | $+\tan \alpha$ | $+\cot \alpha$ | $-\cot \alpha$ | $-\tan \alpha$ | $+\tan \alpha$ |
| cot | $-\cot \alpha$ | $+\tan \alpha$ | $-\tan \alpha$ | $-\cot \alpha$ | $+\cot \alpha$ | $+\tan \alpha$ | $-\tan \alpha$ | $-\cot \alpha$ | $+\cot \alpha$ |
| sec | $+\sec \alpha$ | $+\csc \alpha$ | $-\csc \alpha$ | $-\sec \alpha$ | $-\sec \alpha$ | $-\csc \alpha$ | $+\csc \alpha$ | $+\sec \alpha$ | $+\sec \alpha$ |
| csc | $-\csc \alpha$ | $+\sec \alpha$ | $+\sec \alpha$ | $+\csc \alpha$ | $-\csc \alpha$ | $-\sec \alpha$ | $-\sec \alpha$ | $-\csc \alpha$ | $+\csc \alpha$ |

the angle $90°n+\alpha$ is positive when $\alpha$ is an acute angle, then the signs of both functions are the same; if it is negative, they are different.

The results of the foregoing reduction formulas are summarized in the Table on page 354 where rows for secant and cosecant have been added.

### 194. Addition and Subtraction Formulas

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta,$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta,$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta,$$
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta},$$
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

### 195. Double-Angle, Triple-Angle and Half-Angle Formulas

$$\sin 2\alpha=2\sin\alpha\cos\alpha,$$
$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha=1-2\sin^2\alpha=2\cos^2\alpha-1,$$
$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha};\qquad \cot 2\alpha=\frac{\cot^2\alpha-1}{2\cot\alpha},$$
$$\sin 3\alpha=3\sin\alpha-4\sin^3\alpha;\qquad \cos 3\alpha=4\cos^3\alpha-3\cos\alpha,$$
$$\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha};\qquad \cot 3\alpha=\frac{\cot^3\alpha-3\cot\alpha}{3\cot^2\alpha-1},$$
$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}};\qquad \cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}},$$
$$\tan\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}=\frac{\sin\alpha}{1+\cos\alpha}=\frac{1-\cos\alpha}{\sin\alpha},$$
$$\cot\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}=\frac{\sin\alpha}{1-\cos\alpha}=\frac{1+\cos\alpha}{\sin\alpha}$$

The signs in front of the radicals are taken in accord with the quadrant in which the angle $\frac{\alpha}{2}$ lies (Secs. 192, 193).

**196. Reducing Trigonometric Expressions to Forms Convenient for Taking Logarithms**

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}=2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2},$$

$$\cos\alpha+\sin\alpha=\sqrt{2}\cos(45°-\alpha),$$

$$\cos\alpha-\sin\alpha=\sqrt{2}\sin(45°-\alpha),$$

$$\tan\alpha\pm\tan\beta=\frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}, \qquad \cot\alpha\pm\cot\beta=\frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta},$$

$$\tan\alpha+\cot\beta=\frac{\cos(\alpha-\beta)}{\cos\alpha\sin\beta}; \quad \tan\alpha-\cot\beta=-\frac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta},$$

$$\tan\alpha+\cot\alpha=2\csc 2\alpha; \quad \tan\alpha-\cot\alpha=-2\cot 2\alpha,$$

$$1+\cos\alpha=2\cos^2\frac{\alpha}{2}; \quad 1-\cos\alpha=2\sin^2\frac{\alpha}{2},$$

$$1+\sin\alpha=2\cos^2\left(45°-\frac{\alpha}{2}\right),$$

$$1-\sin\alpha=2\sin^2\left(45°-\frac{\alpha}{2}\right),$$

$$1\pm\tan\alpha=\frac{\sin(45°\pm\alpha)}{\cos 45°\cos\alpha}=\frac{\sqrt{2}\sin(45°\pm\alpha)}{\cos\alpha},$$

$$1\pm\tan\alpha\tan\beta=\frac{\cos(\alpha\mp\beta)}{\cos\alpha\cos\beta}, \quad \cot\alpha\cot\beta\pm 1=\frac{\cos(\alpha\mp\beta)}{\sin\alpha\sin\beta}$$

$$1-\tan^2\alpha=\frac{\cos 2\alpha}{\cos^2\alpha}, \quad 1-\cot^2\alpha=-\frac{\cos 2\alpha}{\sin^2\alpha},$$

$$\tan^2\alpha-\tan^2\beta=\frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\cos^2\alpha\cos^2\beta},$$

$$\cot^2\alpha-\cot^2\beta=\frac{\sin(\alpha+\beta)\sin(\beta-\alpha)}{\sin^2\alpha\sin^2\beta},$$

$$\tan^2\alpha-\sin^2\alpha=\tan^2\alpha\sin^2\alpha, \quad \cot^2\alpha-\cos^2\alpha=\cot^2\alpha\cos^2\alpha$$

### 197. Reducing Expressions Involving the Angles of a Triangle to Logarithmic Form

If $A$, $B$, $C$ are the angles of a triangle or, generally, if $A+B+C=180°$, then certain expressions which do not have logarithmic form may be reduced to logarithmic form by means of the following formulas, which are useful in the solution of oblique triangles.

$$\sin A+\sin B=2\cos\frac{A-B}{2}\cos\frac{C}{2},$$

$$\sin A-\sin B=2\sin\frac{A-B}{2}\sin\frac{C}{2},$$

$$\cos A+\cos B=2\cos\frac{A-B}{2}\sin\frac{C}{2},$$

$$\cos A-\cos B=2\sin\frac{B-A}{2}\cos\frac{C}{2},$$

$$\tan A+\tan B=\frac{\sin C}{\cos A\cos B},$$

$$\cot A+\cot B=\frac{\sin C}{\sin A\sin B},$$

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

$$\tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C,$$

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

### 198. Some Important Relations

$$\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)],$$

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)],$$

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

These formulas may be used to avoid multiplications (they are frequently used in nonlogarithmic computations in higher mathematics; for instance, in the integration of trigonometric functions).

$$\sin\alpha=\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}};\quad \cos\alpha=\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}},$$

$$\tan\alpha=\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

These formulas are useful in the solution of trigonometric equations (and in the integration of trigonometric functions in higher mathematics).

$$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots+\sin n\alpha=\frac{\cos\frac{\alpha}{2}-\cos\frac{(2n+1)\alpha}{2}}{2\sin\frac{\alpha}{2}},$$

$$\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots+\cos n\alpha=\frac{\sin\frac{(2n+1)\alpha}{2}-\sin\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}},$$

$$\cos n\alpha=\cos^n\alpha-C_n^2\cos^{n-2}\alpha\sin^2\alpha+C_n^4\cos^{n-4}\alpha\sin^4\alpha-\ldots,$$

$$\sin n\alpha=n\cos^{n-1}\alpha\sin\alpha-C_n^3\cos^{n-3}\alpha\sin^3\alpha+C_n^5\cos^{n-5}\alpha\sin^5\alpha-\ldots$$

In the last two formulas, $C_n^k$ are binomial coefficients (see Sec. 136). The signs of the terms alternate; the right members break off by themselves in that they terminate in the zeroth or first power of the cosine.

**Examples.**

$$\cos 3\alpha=\cos^3\alpha-3\cos\alpha\sin^2\alpha,$$

$$\sin 3\alpha=3\cos^2\alpha\sin\alpha-\sin^3\alpha,$$

$$\cos 4\alpha=\cos^4\alpha-6\cos^2\alpha\sin^2\alpha+\sin^4\alpha,$$

$$\sin 4\alpha=4\cos^3\alpha\sin\alpha-4\cos\alpha\sin^3\alpha$$

## 199. Basic Relations Between Elements of a Triangle*

Notation: $a$, $b$, $c$ denote the sides of a triangle; $A$, $B$, $C$ the angles; $p=\frac{a+b+c}{2}$ is the semiperimeter; $h$ is the altitude; $S$ stands for area; $R$ denotes the radius of a circumscribed circle; $r$, the radius of an inscribed circle; $r_a$, the radius of a circle tangent to side $a$ and to sides $b$ and $c$ produced (escribed circle); $h_a$ is the altitude drawn to side $a$; $\beta_A$ is the bisector of angle $A$.

(1) Law of cosines:

$$a^2=b^2+c^2-2bc\cos A \quad \text{or} \quad \cos A=\frac{b^2+c^2-a^2}{2bc}$$

(cf. Sec. 147).

(2) From it are derived the half-angle formulas:

$$\sin\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}}, \quad \cos\frac{A}{2}=\sqrt{\frac{p(p-a)}{bc}},$$

$$\tan\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}=\frac{1}{p-a}\sqrt{\frac{(p-a)(p-b)(p-c)}{p}}=\frac{r}{p-a}$$

from which we obtain

$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{p-c}{p}; \quad \frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}=\frac{p-b}{p-a}$$

(3) Law of sines:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

From this law we can derive the following two formulas.

(4) Law of tangents (formula of Regiomontanus):

$$\frac{a+b}{a-b}=\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}=\frac{\cot\frac{C}{2}}{\tan\frac{A-B}{2}}$$

---

* All formulas are given in one version only; two similar formulas may be obtained from each one by a corresponding change of letters. For example, from the formula

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

we get $\cos B=\frac{a^2+c^2-b^2}{2ac}$;

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

(5) Mollweide's equations:

$$\frac{a+b}{c}=\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}\,;\quad \frac{a-b}{c}=\frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}$$

(6) Formulas of area:

$$S=\frac{bc\sin A}{2},\quad S=\frac{b^2\sin A\sin C}{2\sin B},$$

$$S=\sqrt{p(p-a)(p-b)(p-c)},\quad S=\frac{h^2\sin B}{2\sin A\sin C},$$

$$S=r^2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2},\quad S=p^2\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2},$$

$$S=p(p-a)\tan\frac{A}{2},\quad S=\frac{h_a^2\sin A}{2\sin B\sin C},\quad S=\sqrt{rr_ar_br_c}$$

(7) Radii of circumscribed, inscribed and escribed circles:

$$R=\frac{a}{2\sin A}=\frac{abc}{4S}=\frac{p}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{bc}{2h_a},$$

$$4R=r_a+r_b+r_c-r,$$

$$r=\frac{S}{p}=(p-a)\tan\frac{A}{2}=$$

$$=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

$$\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c},$$

$$r_a=\frac{S}{p-a}=p\tan\frac{A}{2}$$

(8) Bisector:

$$\beta_A=\frac{h_a}{\cos\frac{B-C}{2}}$$

### 200. Solving Oblique Triangles

**Case 1.** Given three sides $a$, $b$, $c$.

(a) When using tables of natural functions, first find one of the angles by the law of cosines:

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

The second angle (say $B$) is found by the law of sines:

$$\sin B = \frac{b \sin A}{a}$$

The third angle is found by the formula

$$C = 180° - (A + B)$$

If considerable accuracy is required (even up to 10′), the computation (particularly of the first result) is exceedingly arduous.

(b) When using tables of logarithms, the angles $A$, $B$, $C$ (it is sufficient to compute two of them) are found from one of the half-angle formulas (Sec. 199, Item 2).

**Work:**

$$\text{Given: } a = 74,\ b = 130;\ c = 186.$$

$$2p = a + b + c = 390,\quad p = 195,\quad \log p = 2.2900,$$

| | |
|---|---|
| $p - a = 121$ | $\log (p - a) = 2.0828$ |
| $p - b = 65$ | $\log (p - b) = 1.8129$ |
| $p - c = 9$ | $\log (p - c) = 0.9542$ |

(1) Compute $A$:

$$\tan \frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}},$$

$$\log (p-b) = 1.8129$$
$$\log (p-c) = 0.9542$$
$$\operatorname{colog} p = 3.7100$$
$$\operatorname{colog} (p-a) = \bar{3}.9172$$
$$\overline{2}.3943$$

$$\log \tan \frac{A}{2} = \tfrac{1}{2} \cdot \overline{2}.3943 = \bar{1}.1971$$

$$\frac{A}{2} = 8°57',\quad A = 17°54'$$

(2) Compute $B$:

$$\tan \frac{B}{2} = \sqrt{\frac{(p-a)(p-c)}{p(p-b)}}$$

A similar computation yields

$$B = 32°40'$$

(3) Compute $C$ as a check:

$$\tan\frac{C}{2}=\sqrt{\frac{(p-a)(p-b)}{p(p-c)}}$$

The result is $C=129°26'$.

Check: $A=17°54'$
$B=32°40'$
$C=129°26'$
$A+B+C=180°$

**Case 2.** Given two sides $a$, $b$ and the angle between them, $C$.

(a) When using tables of natural functions, first find side $c$ by the law of cosines:

$$c^2=a^2+b^2-2ab\cos C$$

and then angle $A$ by the law of sines:

$$\sin A=\frac{a\sin C}{c}$$

Here, angle $A$, which corresponds to the sine just found, is acute if $\frac{b}{a}>\cos C$ and obtuse if $\frac{b}{a}<\cos C$.

The third angle is determined either by the formula $C=180°-(A+B)$ or in the same way as $A$ (for a check). Finding side $c$ to a high degree of accuracy involves arduous computations.

(b) When using tables of logarithms, side $c$ is found by the law of sines after angles $A$ and $B$ have been determined. $A$ and $B$ are found by the law of tangents:

$$\frac{a+b}{a-b}=\frac{\cot\frac{C}{2}}{\tan\frac{A-B}{2}}$$

from which, using given $a$, $b$, $C$ we find $\frac{A-B}{2}$ and, since $\frac{A+B}{2}\left(=90°-\frac{C}{2}\right)$ is also known, we readily get $A$ and $B$.

**Work:**

Given: $a=289$ $C=100°19'$.

(1) Compute $\frac{B-A}{2}$:

$$\tan\frac{B-A}{2}=\frac{b-a}{b+a}\cot\frac{C}{2},$$

$$\begin{aligned}\log(b-a)&=2.4942\\ \log\cot\tfrac{C}{2}&=\bar{1}.9214\\ \operatorname{colog}(b+a)&=\bar{3}.0506\\ \log\tan\tfrac{B-A}{2}&=\bar{1}.4662,\quad \tfrac{B-A}{2}=16°18'\end{aligned}$$

(2) Compute $B$ and $A$:

$$\frac{B+A}{2}=90°-\frac{C}{2}=39°50',\quad \frac{B-A}{2}=16°18'$$

Adding we get $B=56°8'$. Subtracting we get $A=23°32'$.

(3) Compute side $c$:

$$c=\frac{a\sin C}{\sin A},$$

$$\begin{aligned}\log a&=2.4609\\ \log\sin C&=\bar{1}.9929\\ \operatorname{colog}\sin A&=0.3987\\ \log c&=2.8525,\quad c=712.0\end{aligned}$$

**Case 3.** Given any two angles (say $A$ and $B$) and side $c$. We carry out the computations in the following manner whether using logarithms or not: first determine the third angle of the triangle by the formula $180°-(A+B)$, then the sides $a$ and $b$ by the law of sines. Using logarithms, write out the work as follows:

Given: $A=55°20'$, $B=44°41'$, $c=795$.

(1) Compute angle $C$: $C=180°-(A+B)=79°59'$.

(2) Compute side $a$:

$$a=\frac{c\sin A}{\sin C},$$

$$\begin{aligned}\log c&=2.9004\\ \log\sin A&=\bar{1}.9151\\ \operatorname{colog}\sin C&=0.0067\\ \log a&=2.8222,\quad a=664.0\end{aligned}$$

(3) Compute side $b$:

Using the formula $b=\frac{c\sin B}{\sin C}$ in the same way as above, we obtain $b=567.7$.

**Case 4.** Given two sides $a$ and $b$ and angle $B$ opposite one of them.

Carry out the computations as follows whether using logarithms or not: first find angle $A$ opposite the other given side by the law of sines: $\sin A = \frac{a \sin B}{b}$. We then get the following possibilities:

(a) $a > b$; $a \sin B > b$ and the problem has no solution;

(b) $a > b$; $a \sin B = b$ and there is one solution: angle $A$ is a right angle;

(c) $a > b$; $a \sin B < b < a$; the problem has two solutions: angle $A$ that corresponds to the computed sine may be taken acute or obtuse;

(d) $a \leqslant b$; the problem has one solution: angle $A$ is taken acute.

Having determined angle $A$, we find $C$ by the formula $C = 180° - (A + B)$. If $A$ can have two values, then two values are obtained for $C$ as well. Finally, the third side $c$ is found by the law of sines, $c = \frac{b \sin C}{\sin B}$. If two values of $C$ are found, then $c$ has two values as well and thus the conditions are satisfied by two distinct triangles.

**Work:**

$$\text{Given: } a = 360.0; \quad b = 309.0; \quad B = 21°14'.$$

We have $a > b$ and $a \sin B < b$ (this is revealed in the first few steps of the computation). Hence we have Case 4*c*.

(1) Compute angle $A$:

$$\sin A = \frac{a \sin B}{b},$$

$$\begin{aligned} \log a &= 2.5563 \\ \log \sin B &= \bar{1}.5589 \\ \operatorname{colog} b &= 3.5100 \\ \hline \log \sin A &= \bar{1}.6252 \end{aligned}$$

If we had $a \sin B > b$, the characteristic of the logarithm would be positive and the problem would have no solution. First solution $A_1 = 24°57'$; second solution $A_2 = 180° - 24°57' = 155°3'$.

(2) Compute angle $C = 180° - (A + B)$:
first solution $C_1 = 133°49'$; second solution $C_2 = 3°43'$.

(3) Compute side $c$:

$$c = \frac{b \sin C}{\sin B}$$

| first solution | second solution |
|---|---|
| $\log b = 2.4900$ | $\log b = 2.4900$ |
| $\log \sin C_1 = \bar{1}.8583$ | $\log \sin C_2 = \bar{2}.8117$ |
| $\operatorname{colog} \sin B_1 = 0.4411$ | $\operatorname{colog} \sin B_2 = 0.4411$ |
| $\log c_1 = 2.7894,\ c_1 = 615.7.$ | $\log c_2 = 1.7428,\ c_2 = 55.31$ |

### 201. Inverse Trigonometric Functions (Circular Functions)

The relation $x = \sin y$ makes it possible, with the aid of tables, to find $x$ if $y$ is known, and $y$ if $x$ is known (and does not exceed 1 in absolute value). Thus, we can consider the sine as a function of an angle, and also the angle as a function of the sine. This fact is apparent in the notation $y = \arcsin x$ (arcsin is pronounced "ark-sine"). For example, in place of $\frac{1}{2} = \sin 30°$ we can write $30° = \arcsin \frac{1}{2}$. In the latter case, the angle is usually expressed in radians and not degrees so that one writes $\frac{\pi}{6} = \arcsin \frac{1}{2}$. Although this notation is simply a variation of the notation $\frac{1}{2} = \sin \frac{\pi}{6}$, the student is often confused at first. Yet the student finds nothing out of the ordinary when writing $2^3 = 8$ and $2 = \sqrt[3]{8}$. This is because the rules for taking roots differ from those for raising to a power, and the student is accustomed to viewing them as two distinct operations, whereas finding the sine from an angle and an angle from the sine is done in the same tables, in which only the term "sine" is used and "arcsine" is not even mentioned. For this reason, the student does not perceive of any specific operation whose result is an arcsine. To put it generally, there is really no reason for introducing this concept in elementary mathematics. In higher mathematics the arcsine occurs often enough as a result of a certain operation called integration; it is precisely here that the concept arcsine originated.

**Definition.** arcsin $x$ means the angle whose sine is $x$. Similar definitions pertain to $\arccos x$, $\arctan x$, $\operatorname{arccot} x$, $\operatorname{arcsec} x$, $\operatorname{arccsc} x$. The functions $\arcsin x$, $\arccos x$, etc. are inverse to the functions $\sin x$, $\cos x$, etc. (see Sec. 208) (just as the function $\sqrt{x}$ is the inverse of $x^2$). Hence their name: *inverse trigonometric functions* (or *circular functions*). All inverse trigonometric functions are multiple valued, which

means that the following holds true for all of them: one value of $x$ is associated with an infinity of values of the function (since an infinite number of angles, say $\alpha$, $180°-\alpha$, $360°+\alpha$, have the same sine).

The *principal value* of $\arcsin x$ is the value which lies between $-\frac{\pi}{2}$ $(-90°)$ and $+\frac{\pi}{2}$ $(+90°)$. Thus, the principal value of $\arcsin\frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$; the principal value of $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ is $-\frac{\pi}{4}$.

The *principal value* of $\arccos x$ is the value which lies between 0 and $\pi$ $(+180°)$. Thus, the principal value of $\arccos\frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$; the principal value of $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ is $+\frac{3}{4}\pi$.

The principal values of $\operatorname{arccot} x$ and $\operatorname{arcsec} x$ (like that of $\arccos x$) lie between 0 and $\pi$. The principal values of $\arctan x$ and $\operatorname{arccsc} x$ (like that of $\arcsin x$) lie between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

**Examples.** The principal values of $\arctan(-1)=-\frac{\pi}{4}$,

$$\operatorname{arccot}\sqrt{3}=+\frac{\pi}{6},\quad \operatorname{arcsec}(-2)=+\frac{2}{3}\pi.$$

If we use the designations $\operatorname{Arcsin} x$, $\operatorname{Arccos} x$ and so on to denote arbitrary values of the corresponding inverse trigonometric functions, and if we retain the designations $\arcsin x$, $\arccos x$, etc., for the principal values, then the relationship between the values of an inverse function and its principal value is given by the following formulas:

$$\operatorname{Arcsin} x=k\pi+(-1)^k\arcsin x, \tag{1}$$
$$\operatorname{Arccos} x=2k\pi\pm\arccos x, \tag{2}$$
$$\operatorname{Arctan} x=k\pi+\arctan x, \tag{3}$$
$$\operatorname{Arccot} x=k\pi+\operatorname{arccot} x \tag{4}$$

where $k$ is any integer, positive, negative or zero.

The graphs of inverse trigonometric functions are given in Sec 213.

**Example 1.** $\operatorname{Arcsin}\frac{1}{2}=k\pi+(-1)^k\arcsin\frac{1}{2}$

$=k\pi+(-1)^k\frac{\pi}{6}$.

For $k=0$ we have $0\cdot\pi+(-1)^0\frac{\pi}{6}=\frac{\pi}{6}$ (or 30°, the principal value);

for $k=1$ we have $1\cdot\pi+(-1)\frac{\pi}{6}=\pi-\frac{\pi}{6}=\frac{5}{6}\pi$ (or 150°);

for $k=2$ we have $2\cdot\pi+(-1)^2\frac{\pi}{6}=2\pi+\frac{\pi}{6}=2\frac{1}{6}\pi$ (or 390°);

for $k=-1$ we have $-\pi+(-1)^{-1}\frac{\pi}{6}=-\pi-\frac{\pi}{6}=-1\frac{1}{6}\pi$ (or −210°);

for $k=-2$ we have $-2\pi+(-1)^{-2}\frac{\pi}{6}=-2\pi+\frac{\pi}{6}=-1\frac{5}{6}\pi$ (or −330°); and so forth.

**Example 2.** $\operatorname{Arccos}\frac{1}{2}=2k\pi\pm\arccos\frac{1}{2}=2k\pi\pm\frac{\pi}{3}$.

For $k=0$ we have $\frac{\pi}{3}$ (or 60°, the principal value) and $-\frac{\pi}{3}$ (or −60°); for $k=1$ we have $2\pi+\frac{\pi}{3}=2\frac{1}{3}\pi$ (or 420°) and $2\pi-\frac{\pi}{3}=1\frac{2}{3}\pi$ (or 300°), and so forth.

### 202. Basic Relations for Inverse Trigonometric Functions*

$$\begin{array}{ll}\sin \operatorname{Arcsin} a=a & \operatorname{Arcsin}(\sin\alpha)=k\pi+(-1)^k\alpha,\\ \cos \operatorname{Arccos} a=a, & \operatorname{Arccos}(\cos\alpha)=2k\pi\pm\alpha,\\ \tan \operatorname{Arctan} a=a, & \operatorname{Arctan}(\tan\alpha)=k\pi+\alpha,\\ \cot \operatorname{Arccot} a=a, & \operatorname{Arccot}(\cot\alpha)=k\pi+\alpha,\end{array}$$

$$\left.\begin{array}{l}\arcsin a=\arccos\sqrt{1-a^2}=\arctan\dfrac{a}{\sqrt{1-a^2}},\\ \arccos a=\arcsin\sqrt{1-a^2}=\operatorname{arccot}\dfrac{a}{\sqrt{1-a^2}},\\ \arctan a=\operatorname{arccot}\dfrac{1}{a}=\arcsin\dfrac{a}{\sqrt{1+a^2}}=\arccos\dfrac{1}{\sqrt{1+a^2}}\end{array}\right\}\text{for } a>0$$

$$\arcsin a+\arccos a=\frac{\pi}{2},$$

$$\arctan a+\operatorname{arccot} a=\frac{\pi}{2},\quad \operatorname{arcsec} a+\operatorname{arccsc} a=\frac{\pi}{2},$$

---

*The roots in all formulas of this section are positive numbers.

$$\operatorname{Arcsin} a + \operatorname{Arcsin} b = \operatorname{Arcsin}\ (a\sqrt{1-b^2} + b\sqrt{1-a^2}),$$

$$\operatorname{Arcsin} a - \operatorname{Arcsin} b = \operatorname{Arcsin}\ (a\sqrt{1-b^2} - b\sqrt{1-a^2}),$$

$$\operatorname{Arccos} a + \operatorname{Arccos} b = \operatorname{Arccos}\ (ab - \sqrt{1-a^2}\sqrt{1-b^2}),$$

$$\operatorname{Arccos} a - \operatorname{Arccos} b = \operatorname{Arccos}\ (ab + \sqrt{1-a^2}\sqrt{1-b^2}),$$

$$\operatorname{Arctan} a + \operatorname{Arctan} b = \operatorname{Arctan} \frac{a+b}{1-ab},$$

$$\operatorname{Arctan} a - \operatorname{Arctan} b = \operatorname{Arctan} \frac{a-b}{1+ab},$$

$$\arcsin a + \arcsin b = \begin{cases} \arcsin\ (a\sqrt{1-b^2} + b\sqrt{1-a^2}) \text{ (if } a^2+b^2 < 1, \text{ also if } a^2+b^2 > 1, \text{ but } ab < 0), \\ \pm\,[\pi - \arcsin\ (a\sqrt{1-b^2} + b\sqrt{1-a^2})] \text{ (if } a^2+b^2 > 1 \text{ and } ab > 0), \end{cases}$$

$$\arcsin a - \arcsin b = \begin{cases} \arcsin\ (a\sqrt{1-b^2} - b\sqrt{1-a^2}) \text{ (if } a^2+b^2 < 1, \text{ also if } a^2+b^2 > 1, \text{ but } ab > 0), \\ \pm\,[\pi - \arcsin\ (a\sqrt{1-b^2} - b\sqrt{1-a^2}] \text{ (if } a^2+b^2 > 1, \text{ but } ab < 0). \end{cases}$$

In the last two formulas take the + sign in front of the square brackets if $a$ is positive and the — sign if $a$ is negative.

### 203. On the Construction of Tables of Trigonometric Functions

The arc of a circle ($\overset{\frown}{MAM_1}$ in Fig. 230) is always longer than the subtending chord ($MPM_1$) so that $\frac{\overset{\frown}{MAM_1}}{MPM_1} > 1$. However, the smaller the central angle $MOM_1$, the less the ratio $\frac{\overset{\frown}{MAM_1}}{MPM_1}$ differs from unity, and hence the smaller the error if we consider the arc and its chord to be equal. Thus, for a central angle of 10°, the arc $MM_1$ amounts to 0.174533 $r$ ($r$ the radius of the circle) and its chord is 0.174312 $r$:

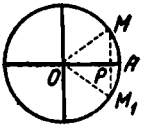

Fig. 230.

$$\frac{0.174533\,r}{0.174312\,r} \approx 1.001$$

By taking the chord equal to the arc, we have an error of 0.0002 $r$, which is only about one tenth of one percent.

For an angle of 2° the relative error is about ten times less, namely, the arc is equal to 0.034907 $r$; the chord is equal to 0.034904 $r$. Their ratio $\frac{0.034907\,r}{0.034904\,r} \approx 1.0001$. Assuming the arc equal to the chord, we have an error of about one hundredth of one percent.

On the other hand, the ratio of the arc $\overset{\frown}{MAM_1}$ to the chord $MPM_1$ is exactly equal to the ratio of the radian measure of angle $MOA$ (which constitutes one half of the angle $MOM_1$) to its sine. Indeed, $\overset{\frown}{MAM_1} : MPM_1 = 2\overset{\frown}{MA} : 2MP = \overset{\frown}{MA} : MP = \frac{\overset{\frown}{MA}}{R} : \frac{MP}{R}$ but $\frac{\overset{\frown}{MA}}{R}$ is the radian measure of the angle $MOA$ (Sec. 180) and $\frac{MP}{R}$ is the sine of the same angle.

This means that by taking for $\sin\alpha$ the value of the angle $\alpha$ itself (in radian measure) we have a small error if the angle $\alpha$ is small. By taking a small enough angle we can find the sine of the angle to the desired degree of accuracy. Then we can construct the entire table of trigonometric functions. Suppose we have found, say, $\sin 30'$. Then by the formula $\cos 30' = \sqrt{1-\sin^2 30'}$ we also find the cosine of this angle; then $\tan 30'$, $\cot 30'$, etc. are found from the formulas on page 351. Continuing, formulas $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ will permit finding $\sin(2\times 30') = \sin 1°$ and $\cos 1°$. Then, using the addition formulas (Sec. 194), we can compute $\sin(1°+30') = \sin 1°30'$ and $\cos(1°+30') = \cos 1°30'$. Now, knowing the sine and cosine of the angles 1°30′ and 30′, we can find $\sin 2°$, $\cos 2°$, etc.

In this way we can construct tables of the trigonometric functions (using this procedure, we first have to find the number $\pi$ to a sufficient degree of accuracy, otherwise we will not obtain the radian measure of the angle). However, the computations are extremely involved. Prior to the 18th century, table makers (Sec. 179) employed computations that were almost as unwieldy as those. Today, mathematicians have much faster methods based on higher mathematics.

### 204. Trigonometric Equations

An equation involving an unknown quantity under the sign of a trigonometric function is called a *trigonometric equation.**

**Example 1.** The equation $\sin y = \frac{1}{2}$ is trigonometric. Its roots are $y = 30°$, $y = 180° - 30° = 150°$, $y = 2 \cdot 180° + 30° = 390°$, $y = 3 \cdot 180° - 30° = 510°$, etc. and also $y = -180° - 30° = -210°$, $y = -2 \cdot 180° + 30° = -330°$, etc.

The general solution (that is, the totality of all roots) may be written thus [cf. Sec. 201, formula (1)]:

$$y = k \cdot 180° + (-1)^k \cdot 30°$$

where $k$ is any integer, positive, negative or zero.

Consider one of the solutions, say $y = 30°$.

It may also be written as $y = 1800'$ or $y = 108{,}000''$ or $y = \frac{\pi}{6} \approx 0.5236$ (we assume the name "radians"). Thus, in the equation $\sin y = \frac{1}{2}$ the unknown $y$ is the size of the angle and not of its numerical measure. The numerical measure depends on the choice of units of angular measurement (degrees, minutes, radians, etc.).

If we take the numerical measure of the angle for the unknown quantity, we have to indicate in what units the angles are measured (see Example 2).

**Example 2.** In Fig. 231, the chord $AK$ is equal to the radius of the circle, $R = OA$. How many degrees are there in the central angle $AOK$?

Here the desired quantity is a *number*; denote it by $x$; then the size of the angle $AOK$ is $x°$ ($\angle AOK = x°$). Constructing the bisector $OD$ of angle $AOK$, we have $\angle AOD = \left(\frac{x}{2}\right)°$. Since $AK = 2AD = 2OA \cdot \sin \angle AOD = 2R \cdot \sin \left(\frac{x}{2}\right)°$ and $AK = R$ (by hypothesis), we get the equation $2R \cdot \sin \left(\frac{x}{2}\right)° = R$,

* Some writers take the term "trigonometric equation" in its narrow sense and demand that the unknown only be under the sign of a trigonometric function. In that sense, the equation of Example 3 is not a trigonometric equation. However, no matter how we regard the term "trigonometric equation", whether involving equations in which the unknown occurs only under the sign of a trigonometric function or in other combinations as well, a consideration of such equations is useful in many respects.

or

$$\sin\left(\frac{x}{2}\right)^\circ = \frac{1}{2}$$

One of the solutions of this equation is $x = 60$.

In school, problems like these are solved where both procedures for setting up the trigonometric equation are equally suitable, and preference is usually given to the first method. However, problems often crop up that cannot be handled by the first method (see Example 3).

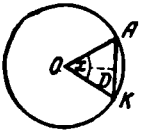

Fig. 231.

**Example 3.** In Fig. 231, arc $AK$ exceeds the chord subtending it by $\frac{\pi}{3} \approx 1.0472$ times. Find the central angle $AOK$.

Apply the second method. Denote by $x$ the degree measure of the desired angle (i. e., $x$ is a *number*).

As in Example 2, find $AK = 2R \sin\left(\frac{x}{2}\right)^\circ$. The degree measure of the arc $\overset{\smile}{AK}$ is also equal to $x$, that is, the length of arc $\overset{\smile}{AK}$ constitutes $\frac{x}{360}$ of the length of the circumference $2\pi R$. Hence

$$\overset{\smile}{AK} = \frac{x}{360} \cdot 2\pi R = \frac{\pi R x}{180}$$

$\overset{\smile}{AK} : AK = \frac{\pi}{3}$ by hypothesis, and we get the equation

$$\frac{\pi R x}{180} : 2R \sin\left(\frac{x}{2}\right)^\circ = \frac{\pi}{3}$$

That is,

$$x : \sin\left(\frac{x}{2}\right)^\circ = 120 \qquad (1)$$

This equation has a unique solution, $x = 60$, which means the desired angle $AOK$ is equal to 60°.

If for the unknown $x$ we took the measure of the angle $AOK$ in minutes, we would get the equation

$$x : \sin\left(\frac{x}{2}\right)' = 7200 \qquad (2)$$

(its root is $x = 3600$, or $\angle AOK = 3600'$).

Thus, by taking a different unit of angle measure, we get a substantially different equation. In other words, in this

specific problem it is not possible to set up an equation in which $x$ denotes the magnitude of the angle and not that of its numerical measure.

**Note.** Denote by $x$ the radian measure of angle $AOK$ and we get the equation

$$x: \sin \frac{x}{2} = \frac{2}{3}\pi \tag{3}$$

$\left(\text{its root is } x = \frac{\pi}{3}\right)$.

In outward appearance, this equation would seem to suggest that $x$ denotes the desired angle $AOK$ rather than its numerical measure. Actually, $x$ here is a number, the radian measure of angle $AOK$, because equation (3) is a contracted form of the equation $x: \sin\left(\frac{x}{2}\right) \text{ radians} = \frac{2}{3}\pi$. In the same way, we could write

$$x: \sin \frac{x}{2} = 120$$

instead of equation (1).

### 205. Techniques for Solving Trigonometric Equations

When solving trigonometric equations, an attempt is made to find the values of some trigonometric function of the unknown. Then, using tables, it is possible to find the values of the unknown itself (which, in the general case, are approximate). The formulas of Sec. 201 serve for working out general solutions.

An equation may be solved in different ways. The formulas given in Sec. 196 and, particularly, in Secs. 194 and 195 may prove useful.

When manipulating trigonometric equations, it is important to take care that the transformed equation is equivalent to the original one. Incidentally, it is sometimes advisable to perform transformations in which equivalence cannot be guaranteed beforehand. But then, in the case of possible extraneous roots appearing (say when squaring both members of the equation; see Examples 5 and 6), it is necessary to check all the solutions found. In the case of a loss of roots, establish which specific roots could be lost and whether they have actually been lost.

Incidentally, it is easy to avoid the danger of losing roots. An example will suffice to illustrate this. Suppose we have

the equation $\tan x = 2 \sin x$. Write it as $\frac{\sin x}{\cos x} = 2 \sin x$. Dividing both members by $\sin x$, we get the equation $\frac{1}{\cos x} = 2$, which is not equivalent to the original one: the lost roots of the equation are $\sin x = 0$. But we can proceed differently. Transpose $2 \sin x$ to the left and factor out $\sin x$ to get $\sin x \left(\frac{1}{\cos x} - 2\right) = 0$, an equivalent equation. It is satisfied in only two cases: (1) if $\sin x = 0$, (2) if $\frac{1}{\cos x} = 2$, that is, $\cos x = \frac{1}{2}$. In the former case, $x = k\pi$; in the latter, $x = 2k\pi \pm \frac{\pi}{3}$. We have all the roots.

**Note.** When equating one of the factors to zero, take care that the other factor *does not become infinite*. In our example, for $\sin x = 0$, we have $\cos x = \pm 1$ so that $\frac{1}{\cos x} - 2$ is equal to $-1$ or $-3$. When $\cos x = \frac{1}{2}$ we have $\sin x = \pm \frac{\sqrt{3}}{2}$. But if the second factor becomes infinite, the result will, as a rule, be incorrect. Suppose we have the equation $\sin x = 0$. We can write, equivalently, $\cos x \cdot \tan x = 0$, but we cannot put $\cos x = 0$ (for $\cos x = 0$, the equation $\sin x = 0$ is definitely not satisfied). The source of the error lies in the fact that for $\cos x = 0$ the function $\tan x$ becomes infinite $\left(\tan x = \frac{\pm \sqrt{1 - \cos^2 x}}{\cos x}\right)$.

The simplest in conception but not always the shortest procedure for solving a trigonometric equation is this: all trigonometric functions involved in the equation are expressed in terms of one and the same function of one and the same quantity, say in terms of $\sin x$ or $\tan x$ or $\tan \frac{x}{2}$, etc. (the table on page 351 and the formulas for $\sin \alpha$, $\cos \alpha$, $\tan \alpha$ in Sec. 201). An apt choice of this function often reduces computational work.

**Example 1.** $3 + 2 \cos \alpha = 4 \sin^2 \alpha$.

It is convenient here to express $\sin^2 \alpha$ in terms of $\cos \alpha$. We have $\sin^2 \alpha = 1 - \cos^2 \alpha$, and we get an equivalent equation:

$$3 + 2 \cos \alpha = 4 (1 - \cos^2 \alpha) \text{ or } 4 \cos^2 \alpha + 2 \cos \alpha - 1 = 0$$

This is a quadratic equation in $\cos \alpha$. We find two values of $\cos \alpha$:

$$(\cos \alpha)_1 = \frac{-1 + \sqrt{5}}{4} = 0.3090, \quad (\cos \alpha)_2 = \frac{-1 - \sqrt{5}}{4} = -0.8090$$

whence $\alpha = 360° k \pm 72°00'$ and $\alpha = 360° k \pm 144°00'$.

**Example 2.** $\frac{3}{\cos^2 x} = 8 \tan x - 2$.

Here, it is convenient to express $\cos^2 x$ in terms of $\tan x$. We have $\cos^2 x = \frac{1}{1+\tan^2 x}$ and obtain the equivalent equation

$$3 \tan^2 x - 8 \tan x + 5 = 0$$

Whence $(\tan x)_1 = 1$, $(\tan x)_2 = \frac{5}{3}$. The equation has the solutions: $x = 180° k + 45°$ and $x = 180° k + 59°02'$ (the first formula is exact, the second, approximate).

**Example 3.** $\sin^2 x - 5 \sin x \cos x - 6 \cos^2 x = 0$.

The simplest procedure here is to divide by $\cos^2 x$. We get

$$\tan^2 x - 5 \tan x - 6 = 0$$

We do not lose roots in dividing by $\cos x$. Indeed, putting $\cos x = 0$ in the given equation, we find $\sin x = 0$, and the equations $\cos x = 0$ and $\sin x = 0$ are inconsistent.

From the equation $\tan^2 x - 5 \tan x - 6$ we find $(\tan x)_1 = 6$ and $(\tan x)_2 = -1$. The roots are $x = 80°32' + 180° k$ and $x = -45° + 180° k$.

**Example 4.** $2 \sin^2 x + 14 \sin x \cos x + 50 \cos^2 x = 26$.

Here it is not advisable to express $\cos x$ in terms of $\sin x$ or vice versa since an irrational expression appears in the second term. Rationalization is possible by isolating the term and squaring, but that is complicated and, what is more, extraneous solutions may appear. It will be better to express $\sin x$ and $\cos x$ in terms of $\tan x$. We have

$$\sin x = \frac{\tan x}{\pm\sqrt{1+\tan^2 x}} \qquad \cos x = \frac{1}{\pm\sqrt{1+\tan^2 x}}$$

In these formulas, take both upper signs or both lower signs (since $\sin x : \cos x$ must be equal to $\tan x$ and not to $-\tan x$). We get the equivalent equation

$$\frac{2 \tan^2 x + 14 \tan x + 50}{1+\tan^2 x} = 26$$

Clear fractions. There will be no extraneous roots since $1+\tan^2 x$ cannot be equal to zero. Collecting like terms, we get the equivalent equation *

$$24 \tan^2 x - 14 \tan x - 24 = 0$$

* This equation can be obtained faster by using the following artificial device since $\sin^2 x + \cos^2 x = 1$, the right member of the given equation may be written as $26 (\sin^2 x + \cos^2 x)$. Then transpose all terms to the left and divide by $\cos^2 x$.

Whence $(\tan x)_1 = \frac{4}{3}$ and $(\tan x)_2 = -\frac{3}{4}$.

The solutions are: $x = 53°07' + 180° k$; $x = -36°52' + 180° k$.

**Example 5.**

$$\sin x + 7 \cos x = 5 \tag{1}$$

Express $\sin x$ in terms of $\cos x$:

$$\pm \sqrt{1 - \cos^2 x} + 7 \cos x = 5 \tag{2}$$

or

$$\pm \sqrt{1 - \cos^2 x} = 5 - 7 \cos x$$

If the values of $\cos x$ were known, we would know what sign to put in front of the radical (plus if the right member is positive, minus if it is negative). We have to keep both signs since we do not know the roots of (1). Therefore, equation (2) is not equivalent to (1). We have introduced extraneous roots. Squaring both members of (2) and collecting terms, we get the equation

$$50 \cos^2 x - 70 \cos x + 24 = 0 \tag{3}$$

which is equivalent to (2) but not to (1).

We find $(\cos x)_1 = 0.8$, $(\cos x)_2 = 0.6$.

Whence $x = \pm 36°52' + 360° k$ and $x = \pm 53°07' + 360° k$.

Check the roots obtained. Substituting $\cos x = 0.8$ into (1), we get $\sin x = 5 - 7 \cos x = 5 - 5.6 = -0.6$. Hence the roots $x = +36°52' + 360° k$ are extraneous since the sines of these angles (they lie in the first quadrant) are equal to $+0.6$. Now the roots $-36°52' + 360° k$ are also those of equation (1) since the sines of these angles equal $-0.6$.

Now put $\cos x = 0.6$ into (1). We get $\sin x = 0.8$, whence we conclude that the roots $x = +53°07' + 360° k$ are those of (1) as well (the sines of these angles are 0.8), while the roots $x = -53°07' + 360° k$ are extraneous (the sines of these angles are $-0.8$).

The solutions of equation (1) are *

$$x = -36°52' + 360° k \text{ and } x = 53°07' + 360° k$$

**Example 6.** The equation considered in Example 5 is a special case of the equation $a \sin x + b \cos x = c$. All equations

* Eq. (1) may be written equivalently as $\sin x = 5 - 7 \cos x$. Squaring, we get $\sin^2 x = (5 - 7 \cos x)^2$, but this equation is not equivalent to (1) since it is also obtainable from $-\sin x = 5 - 7 \cos x$. Replacing $\sin^2 x$ by $1 - \cos^2 x$, we again get (3), and the rest of the solution coincides with that given in the text.

of this general type may be solved by the indicated procedure. We now give two other methods using the same example:

$$\sin x + 7\cos x = 5 \qquad (1)$$

**First method.** Square (this introduces extraneous roots; see footnote on page 374) to get

$$\sin^2 x + 14 \sin x \cos x + 49 \cos^2 x = 25$$

Employing one of the devices indicated in Example 4, we get $24\tan^2 x - 14\tan x - 24 = 0$, which is the same equation obtained in Example 4. We again find $(\tan x)_1 = \frac{4}{3}$, $(\tan x)_2 = -\frac{3}{4}$. But this time we have to eliminate the extraneous roots from $x = 53°07' + 180°k$ and $x = -36°52' + 180°k$. If $\tan x = \frac{4}{3}$ we either have $\sin x = 0.8$, $\cos x = 0.6$ or $\sin x = -0.8$, $\cos x = -0.6$. Substituting into (1) we see that only the first pair of values fit, i. e., angle $x$ lies in the first quadrant. Hence, of the roots $x = 53°07' + 180°k$ only those obtained for even values of $k$ are suitable. Putting $k = 2k'$ we get $x = 53°07' + 360°k'$. In the same way we find that of the roots $x = -36°52' + 180°k$ only those are suitable for which $k$ is even, that is,

$$x = -36°52' + 360°k'$$

**Second method.** Express $\sin x$ and $\cos x$ in terms of $\tan \frac{x}{2}$ (formulas of Sec. 198). Simplifying we obtain an equivalent equation: $12\tan^2 \frac{x}{2} - 2\tan \frac{x}{2} - 2 = 0$, whence

$$\left(\tan \frac{x}{2}\right)_1 = \frac{1}{2}, \quad \left(\tan \frac{x}{2}\right)_2 = -\frac{1}{3}$$

We find $\frac{x}{2} \approx 26°34' + 180°k$ and $\frac{x}{2} \approx -18°26' + 180°k$. The roots are $x \approx 53°08' + 360°k$ and $x \approx -36°52' + 360°k$. The advantage of this method is that it does not introduce extraneous roots.

**Note.** The second method has greater generality. When a trigonometric equation is reduced to a form which involves only trigonometric functions of the same angle, then all these functions may, with the aid of the formulas of Sec. 198, be expressed in terms of the tangent of half an angle. In this method, the computations are often more complicated but we dispense with seeking artificial devices and in many cases avoid extraneous roots.

# FUNCTIONS AND GRAPHS

## 206. Constants and Variables

The application of mathematics to the study of natural laws and to their use in technology and engineering made it necessary to introduce the concept of a variable quantity and, by contrast, that of a constant quantity. A *variable* is a quantity which, within the framework of a given problem, takes on various values. A *constant* is a quantity which, within the framework of a given problem, remains unchanged. The same quantity may be a constant in one problem and a variable in another.

**Example.** The boiling temperature $T$ of water is a constant ($T = 100°C$) in most physical problems. However $T$ is a variable whenever we have to consider variations in atmospheric pressure.

This distinction between constant and variable quantities is a frequent feature of higher mathematics; in elementary mathematics the chief distinction is between knowns and unknowns. The unknown quantity is retained in higher mathematics but it does not play the chief role there.

Variables are mostly denoted by the last letters of the alphabet: $x, y, z$, constants, by the first letters, $a, b, c, \ldots$.

## 207. The Functional Relation Between Two Variables

We say that two variable quantities $x$ and $y$ are connected by a *functional relation* if with each value that one quantity can take is associated one or several definite values of the other.

**Example 1.** The boiling point of water (temperature $T$) and the atmospheric pressure $p$ are connected by a functional relation because every value $T$ is associated with one definite value of $p$, and conversely. Thus, if $T = 100°C$, then $p$ is invariably equal to 760 mm of mercury; if $T = 70°C$ then $p = 234$ mm, etc. By contrast, the atmospheric pressure $p$ and the relative humidity of the air $x$ (if regarded as vari-

ables) are not connected by a functional relation: if it is known that $x=90\%$, nothing definite can be said about $p$.

**Example 2.** The area $S$ and the perimeter $p$ of an equilateral triangle are connected by a functional relation. The formula $S=(\sqrt{3}:36)\,p^2$ expresses that relationship.

If it is desirable to emphasize that in a given problem the values of the variable $y$ are to be sought when the values of the variable $x$ are known, then $x$ is called the *independent variable* (or the *argument*) and $y$ is termed the *dependent variable* (or the *function*).

**Example 3.** If, knowing the perimeter $p$ of an equilateral triangle, we wish to make statements concerning the area $S$ (see Example 2), then $p$ is the argument (independent variable) and $S$ is the function (dependent variable).

More often than not $x$ represents the independent variable.

If every value of the argument $x$ is associated with only one value of the function $y$, then the function is called a *one-valued* (single-valued) function, if the association is with two or more values of $y$, then it is a *multiple-valued* (two-valued, three-valued, etc.) function.

**Example 4.** An object is thrown upwards; $s$ is the height it reaches above the earth and $t$ is the time that elapses from the initial instant of flight. The quantity $s$ is a one-valued function of $t$ since at each instant the altitude of the body is a definite quantity. The quantity $t$ is a two-valued function of $s$ since the body is twice at any given altitude: once in the upward flight and once again in the downward fall.

The formula $s=v_0t-\frac{1}{2}gt^2$ which relates the variables $s$ and $t$ (the initial velocity $v_0$ and the acceleration of the earth's gravitation $g$ are constant values here) shows that for a given $t$ we have one value of $s$ and for a given $s$ we have two values of $t$ as defined by the quadratic equation

$$\frac{1}{2}gt^2-v_0t+s=0$$

### 208. The Inverse Function

In describing a function it is quite inessential what letter is used to denote the function and the argument; say, if we have $y=x^2$ and $u=v^2$, then $y$ is the same function of $x$ as $u$ is of $v$; in other words, $x^2$ and $v^2$ represent the very same function, although the arguments differ.

If we interchange the argument and the function in a given functional relation, we obtain a new function, the inverse of the original function.

**Example 1.** Suppose we have a function $u$ of an argument $v$:

$$u=v^2$$

If we interchange argument and function, the quantity $v$ will be a function of $u$ and will be given by the formula $v=\sqrt{u}$. If the argument is denoted in both cases by the same letter $x$, then the original function is $x^2$ and the inverse function is $\sqrt{x}$.

**Example 2.** The inverse function of $\sin x$ is $\arcsin x$. Indeed, if $y=\sin x$, then $x=\arcsin y$ (Sec. 201).

The graph of an inverse function is given in Sec. 213, Item 7.

## 209. Representation of a Function by Formula and Table

Many functional relations may be represented (exactly or approximately) by simple formulas. For example, the relation between the area $S$ of a circle and the radius $r$ is given by the formula $S=\pi r^2$; the relation between the altitude $s$ of a body thrown upward and the time $t$ that elapses from the initial instant is given by the formula $s=v_0t-\frac{1}{2}gt^2$; actually, the latter formula is an approximate one since it disregards both air resistance and the diminution of terrestrial gravitation with increasing altitude.

It often happens that a functional relation cannot be represented as a formula or, if a formula exists, computations prove to be too complicated. In such cases, other modes of representing functions can be used. Such frequently used modes are tables and graphs (see Sec. 212).

**Example.** The functional relation between pressure $p$ and the boiling temperature $T$ of water (cf. Sec. 207, Example 1) cannot be expressed as a single formula capable of giving sufficient accuracy for all cases of practical importance. The relationship can however be given in tabular form (a portion of such a table is illustrated below):

| p, mm | 300 | 350 | 400 | 450 | 500 | 550 | 600 | 650 | 700 |
|---|---|---|---|---|---|---|---|---|---|
| T°C | 75.8 | 79.6 | 83.0 | 85.8 | 88.5 | 91.2 | 93 5 | 95.7 | 97.6 |

For convenience of computation, the values of one variable are mostly taken at regular intervals; this variable is then called the *argument* of the table.

No table of course can contain all values of the argument, but a table of practical utility must contain sufficient values of the argument so that any other value of the function may be obtained to the required degree of accuracy by means of interpolation (see Sec. 64).

**210. Functional Notation**

Suppose that a variable $y$ is some function of a variable $x$. It is immaterial how the function has been specified, whether by formula, table or in some other fashion. The function may not even be known at all; all that is necessary is to establish the very fact of a functional relationship (Sec. 207). This bare fact is denoted as $y=f(x)$.

The letter $f$ (from the Latin functio) does not represent any quantity as such, just as the letters in log, tan, etc. in the notations $\log x$, $\tan x$, etc. Notations like $y=\log x$, $y=\tan x$, etc. are very definite functional relations between $y$ and $x$; the notation $y=f(x)$ represents any functional relation.

If we wish to stress the fact that the functional relation between $z$ and $t$ differs from that between $y$ and $x$, then we take a different letter, say $F$, and write: $z=F(t)$, $y=f(x)$.

Now if we wish to state that the functional relation between $z$ and $t$ is the same as that between $y$ and $x$, then we use the same letter $f$, and we write: $z=f(t)$, $y=f(x)$.

If an expression is given (or has been found) of $y$ in terms of $x$, then we connect the expression with $f(x)$ by an equals sign.

**Examples.** (1) If it is known that $y=x^2$, then we write $f(x)=x^2$

(2) If it is known that $y=\sin x$, then we can write $f(x)=\sin x$.

(3) If $f(x)=\log x$, then the symbol $f(y)$ means $\log y$.

(4) If $f(x)=\sqrt{1+x^2}$ and $F(x)=3x$, then we can write $F(x)f(x)=3x\sqrt{1+x^2}$, $\dfrac{F(y)}{f(z)}=\dfrac{3y}{\sqrt{1+z^2}}$.

## 211. Coordinates

Two mutually rectangular straight lines $XX'$ and $YY'$ (Fig. 232) form a *rectangular coordinate system*. The straight lines $XX'$ and $YY'$ are called the *coordinate axes*, one of which (usually $XX'$) is given horizontally and is called the *axis of abscissas* (x-axis); the other, $YY'$, is the *axis of ordinates* (y-axis); the point $O$ at their intersection is the *origin of coordinates*. A scale unit is chosen arbitrarily for each axis.

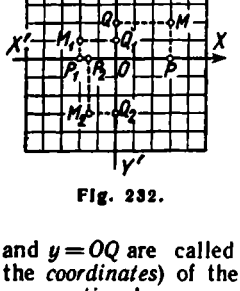

Fig. 232.

Taking an arbitrary point $M$ in a plane with axes, we find its projections $P$ and $Q$ on the axes. The line-segment $OP$ on the axis of abscissas, and also the number $x$ which measures it by the chosen unit of length, or scale unit, is termed the *abscissa* of the point $M$. The segment $OQ$ on the axis of ordinates, and also the number $y$ that measures it, is the *ordinate* of the point $M$. The quantities $x=OP$ and $y=OQ$ are called the *rectangular coordinates* (or, simply, the *coordinates*) of the point $M$. They are considered positive or negative by convention (ordinarily, positive segments are laid off to the right on the axis of abscissas and upwards on the axis of ordinates).

In Fig. 232, where the scales on both axes are the same, point $M$ has abscissa $x=3$ and ordinate $y=2$; point $M_1$ has abscissa $x_1=-2$ and ordinate $y_1=1$. We can shorten this notation to $M(3, 2)$, $M_1(-2, 1)$. Similarly, $M_2(-1.5, -3)$.

Every point in the plane is associated with one number pair: $x$, $y$. Every pair of real numbers $x$, $y$ is associated with one point $M$. A rectangular coordinate system is often called a Cartesian system of coordinates after the French philosopher and mathematician Descartes (which in Latin is Cartesius) who made extensive use of coordinates in the investigation of many geometric problems. This is a misnomer however.*

* Descartes used one axis (not two) on which he laid off abscissas; ordinates were determined as distances of points in the plane from the axis of abscissas; Descartes reckoned these distances in any chosen direction, not necessarily perpendicular. In the hands of Descartes, both abscissas and ordinates were positive irrespective of the

### 212. Graphical Representation of Functions

To depict a given functional relation graphically, mark on the axis of abscissas a number of values $x_1$, $x_2$, $x_3$, ... of one of the variables $x$ (ordinarily the argument), and construct the ordinates $y_1$, $y_2$, $y_3$, ... which are the corresponding values of the other variable $y$ (the function); we thus obtain a number of points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$, .... Join them by a smooth curve drawn free-hand to get the graph of the given functional relation. The advantage of a graphical representation compared to tables lies in its pictorialness and surveyability. A disadvantage is its low accuracy. A proper choice of scales is very important here.

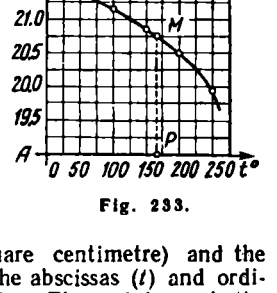

Fig. 233.

Figure 233 gives a graphical representation of a functional relation between the modulus of elasticity $E$ of forged steel (in tons per square centimetre) and the temperature $t$ of iron. The scales of the abscissas ($t$) and ordinates ($E$) are indicated by numerals. (The origin and the axis of abscissas are not shown in the drawing so as not to enlarge the size of the graph unduly).

The graph in Fig. 233 was constructed on the basis of the following table:

| $t°$ (C) | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| $E$ (t/cm²) | 21.5 | 21.4 | 21.2 | 20.9 | 20.5 | 19.9 |

From the graph we can find the approximate values of the function for those values of the argument which are not indicated in the table. For example, suppose it is required to find the value of $E$ for $t = 170°$ C. Lay off on the axis of abscissas (or on the straight line $At$ which is parallel to it) the abscissa $t = AP = 170$ and, erecting the perpendi-

directions of the line-segments. In most textbooks the distinction of direction by the signs + and − is erroneously credited to Descartes, whereas this convention was introduced by his pupils.

cular $PM$, read off the ordinate $E=PM=20.75$. Squared paper facilitates reading a graph. Finding intermediate values of a function from its graph is called graphical interpolation.

A graph is actually constructed (plotted) from points, and then a smooth curve is drawn free-hand connecting the separate points $M_1$, $M_2$, .... Theoretically, it is always possible that the intermediate points not designated in the graph lie far away from the smooth curve. One should therefore define a graph theoretically as the locus of points $M(x, y)$ (Sec. 151) whose coordinates are connected by a given functional relation.

### 213. Elementary Functions and Their Graphs

**1. Proportional quantities.** If variables $y$ and $x$ are directly proportional (Sec. 63), then the functional relationship between them is given by the equation

$$y=mx \qquad (1)$$

where $m$ is a constant (the *proportionality factor*, or *constant of proportionality*, or *constant of variation*). The graph of direct proportionality, or direct variation (here and henceforth it is assumed that the scales on both axes are the

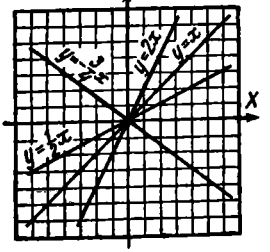


Fig. 234.

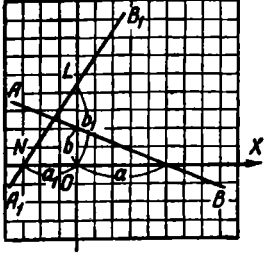


Fig. 235.

same), is a straight line that passes through the origin and forms with the axis of abscissas an angle $\alpha$, the tangent of which is equal to the constant $m$; $\tan\alpha=m$. Therefore the proportionality factor $m$ is also called the *slope of the line*.

Figure 234 shows the graphs of the function $y=mx$ for $m=\frac{1}{2}$, $m=1$, $m=2$, and $m=-\frac{3}{4}$. In determining the angle $\alpha$ between the axis of abscissas and the graph, we reckon counterclockwise from the positive axis of abscissas. The slope of the graph is the tangent of the smallest positive angle through which the positive axis of abscissas can be revolved in order to be parallel to the given line.

**2. Linear function.** If the variables $x$ and $y$ are connected by a first degree equation:

$$Ax+By=C \tag{2}$$

where at least one of the numbers $A$, $B$ is not zero, then the graph of the functional relation is a straight line. When $C=0$, it passes through the origin (cf. Item 1), otherwise it does not.

Suppose that neither $A$ nor $B$ are equal to zero; then the graph intersects both coordinate axes intercepting on the axis of abscissas a segment $a=\frac{C}{A}$ and on the axis of ordinates a segment $b=\frac{C}{B}$.

**Examples.** The graph of the equation $2x+5y=10$ is a straight line $AB$ (Fig. 235); $a=\frac{10}{2}=5$, $b=\frac{10}{5}=2$. The graph of the equation $2y-3x=9$ is a straight line, $A_1B_1$; here, $a_1=\frac{9}{-3}=-3$, $b_1=\frac{9}{2}=4.5$.

Solving (2) for $y$, we get

$$y=mx+b \tag{3}$$

where

$$m=-\frac{A}{B},\quad b=\frac{C}{B}$$

The function $y=mx+b$ is called a *linear function*. Its graph is a straight line.

**Example.** Given the equation $2y-3x=9$. Solving for $y$, we have $y=\frac{3}{2}x+\frac{9}{2}$ $\left(m=-\frac{-3}{2}=\frac{3}{2},\ b=\frac{9}{2}\right)$. The graph of the function $y=\frac{3}{2}x+\frac{9}{2}$ is a straight line, $A_1B_1$ (Fig. 235).

The straight line which serves as the graph of the function $y=mx+b$ forms with the (positively directed) axis of abscissas an angle whose tangent is $m$ and intercepts on the axis of ordinates a segment $b$. The constant $m$ is called the *slope of the line*.

**Example.** For the straight line $A_1B_1$, which is the graph of the function $y=\frac{3}{2}x+\frac{9}{2}$, we have $\tan \angle XNB_1=\frac{3}{2}$, $OL=\frac{9}{2}$.

The equation $y=mx$ (direct proportionality; see Item 1) is a special case of the equation $y=mx+b$ $(b=0)$.

The equation $y=b$ is also a special case of the equation $y=mx+b$ $(m=0)$. In this case the quantity $y$ is constant and hence does not depend on $x$. Still and all, we can consider it a function of the variable $x$, since to each value of $x$ there corresponds a definite value of $y$. The only distinction is that now it has the same value for all values of $x$.

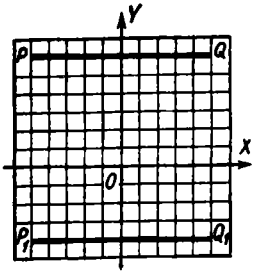


Fig. 236.

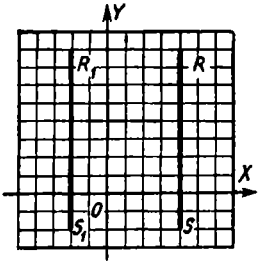


Fig. 237.

The peculiarity of the function $y=b$ $(y=0\cdot x+b)$ lies in the fact that $x$ is not a function of $y$ any longer (because the values of $y$ not equal to $b$ are not associated with any value of $x$). The graph of the function $y=b$ is a straight line parallel to the axis of abscissas.

In Fig. 236 the line $PQ$ is the graph of the equation $y=6$, and $P_1Q_1$ is the graph of the equation $y=-4$.

The equation $y=b$ is derived from (2) when $A=0$ $\left(b=-\frac{C}{B}\right)$. But if $B=0$, then (2) may be given as $x=a$ $\left(a=-\frac{C}{A}\right)$, or $x$

is a constant. It may be taken as a function of the variable $y$ (but $y$ will not be a function of $x$; see above).

The graph of $x=a$ is a straight line parallel to the axis of ordinates. In Fig. 237 the straight line $RS$ is the graph of the equation $x=+4$, while $R_1S_1$ is the graph of the equation $x=-2$.

The axis of abscissas is the graph of the equation $y=0$; the axis of ordinates is the graph of the equation $x=0$.

**3. Inverse proportionality.** If the quantities $x$ and $y$ are inversely proportional (Sec. 63), then the functional relationship between them is expressed by the equation $y=\frac{c}{x}$, where $c$ is a constant. The graph of inverse proportionality is a curve consisting of two branches; for instance, the function $y=\frac{4}{x}$ is given in Fig. 238 as a curve whose branches are $AB$ and $A'B'$. Figure 238 also depicts the graphs of the function $y=\frac{c}{x}$ with $c=1$ (dashed line) and $c=-1$. These curves are called *equilateral hyperbolas* (a plane cutting both nappes of a right circular cone parallel to the axis produces an equilateral hyperbola; see Sec. 167).

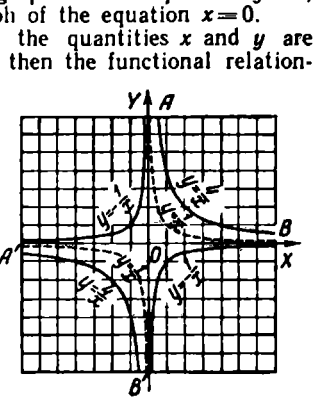

Fig. 238.

**4. Quadratic function.** The function

$$y=ax^2+bx+c$$

where $a$, $b$, $c$ are constants and $a\neq 0$ is called a quadratic function. In the simplest case, $y=ax^2$ ($b=c=0$), the graph is a curve passing through the origin.

In Fig. 239 are depicted graphs of the function $y=ax^2$:

$$AOB\left(a=\frac{1}{2}\right);\ COD\,(a=1);\ EOF\,(a=2);\ KOL\left(a=-\frac{1}{2}\right).$$

The curve (graph) of the function $y=ax^2$ is a parabola (Sec. 167). Every parabola has an axis of symmetry ($OY$ in Fig. 239) called the *axis of the parabola*. The point $O$ of intersection of the parabola with its axis is termed the *vertex of the parabola*.

The graph of the function $y=ax^2+bx+c$ has the same shape as that of the function $y=ax^2$ (for the same value of $a$), hence, it is also a parabola. As before, the axis of this parabola is vertical, but the vertex lies at the point $\left(-\frac{b}{2a}, c-\frac{b^2}{4a}\right)$ and not at the origin.

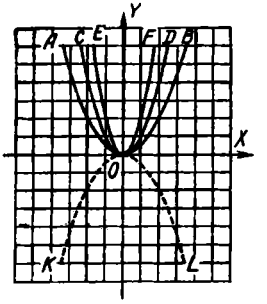


Fig. 239.

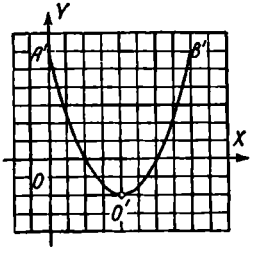


Fig. 240.

**Example.** The graph of the function

$$y=\frac{1}{2}x^2-4x+6$$

where $a=\frac{1}{2}$, $b=4$, $c=6$ is the parabola $A'O'B'$ (Fig. 240) having the same shape as the parabola $y=\frac{1}{2}x^2$ ($AOB$ in Fig. 239). The vertex lies in the point $O'(4, -2)$

$$\left(-\frac{b}{2a}=\frac{4}{2\cdot\frac{1}{2}}=4, \quad c-\frac{b^2}{4a}=6-\frac{16}{4\cdot\frac{1}{2}}=-2\right).$$

**5. Power function.** The function $y=ax^n$ ($a$, $n$ constants) is called a *power function*. The functions $y=ax$, $y=ax^2$, $y=\frac{a}{x}$ (see Items 1, 3, 4) are special cases of a power function ($n=1$, $n=2$, $n=-1$).

Since the zeroth power of any nonzero number is unity,*

* The expression $0^0$ is indeterminate; in the given case, when the function $y=ax^0$ is equal to $a$ for all values of $x$ except zero, we agree that $y$ is equal to $a$ for $x=0$.

the power function for $n=0$ becomes a constant quantity: $y=a$. In this case the graph is a straight line parallel to the axis of abscissas (see Item 2).

The other cases may be split up into two groups: (a) $n$ is a positive number, and (b) $n$ is a negative number.

(a) In Fig. 241 we have the graphs of the function $y=x^n$ for $n=0.1, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}, 2, 3, 4, 10$. They all pass through the origin and through the point (1, 1). For $n=1$ we have a straight line, the bisector of the angle $XOY$. For $n>1$ the graph first goes below the straight line (between $x=0$ and $x=1$) and then above it (for $x>1$); for $n<1$, vice versa.

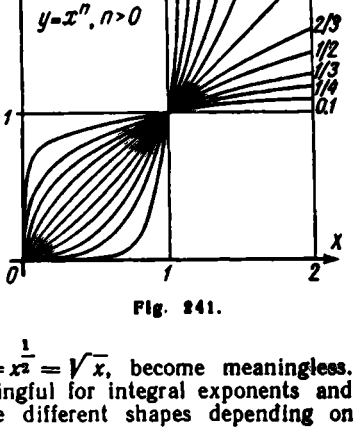

Fig. 241.

We confined ourselves to the case $a=1$ since the other cases are obtained by a simple change in the scale. Negative values of $x$ are not taken since for $x<0$ certain power functions with fractional exponents, say $y=x^{\frac{1}{2}}=\sqrt{x}$, become meaningless. Power functions are meaningful for integral exponents and $x<0$ but the graphs have different shapes depending on whether $n$ is even or odd.

Typical examples are given in Fig. 242: graphs of the functions $y=x^2$ and $y=x^3$. For $n$ even, the graph is symmetric (see Sec. 175) about the axis of ordinates; for $n$ odd, it is symmetric about the origin.

By analogy with the graph of the function $y=ax^2$, the graphs of all power functions $y=ax^n$, for positive $n$, are called *parabolas of order n* (or *degree n*) Thus, the graph of the function $y=ax^3$ (Fig. 242) is a parabola of third order, or a cubical parabola.

**Note.** If $n$ is a fraction $\frac{p}{q}$ with even denominator $q$ and odd numerator $p$, then the quantity $x^n=\sqrt[q]{x^p}$ can have two signs $\left(\pm\sqrt[q]{x^p}\right)$ and the graph will exhibit another part

below the axis of abscissas symmetric to the upper half. In Fig. 243 we have the graph of a two-valued function $y=\pm 2x^{\frac{1}{2}}$, i.e., $x=\frac{1}{4}y^2$ (a parabola with horizontal axis); Fig. 244 shows the graph of a two-valued function $y=\pm\frac{1}{2}x^{\frac{3}{2}}$ *(semicubical parabola, or Neil's parabola)*.

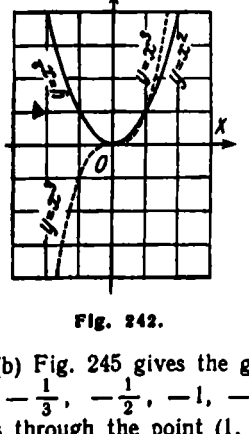


Fig. 242.

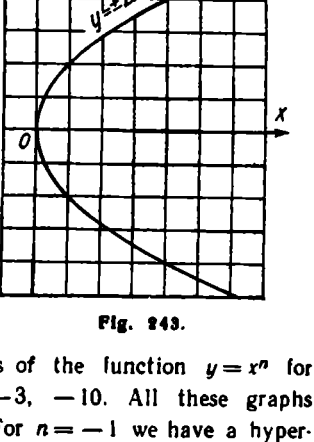


Fig. 243.

(b) Fig. 245 gives the graphs of the function $y=x^n$ for $n=-\frac{1}{3}$, $-\frac{1}{2}$, $-1$, $-2$, $-3$, $-10$. All these graphs pass through the point (1, 1). For $n=-1$ we have a hyperbola (Item 3). For $n<-1$ the graph of the power function at first (between $x=0$ and $x=1$) lies above the hyperbola and then (for $x>1$) below it; for $n>-1$, vice versa. With regard to negative values of $x$ and fractional values of $n$ we can repeat what was said in Subitem (a).

All the graphs of Fig. 245 approach without bound the axis of abscissas and the axis of ordinates without actually reaching either. Because they resemble hyperbolas, these graphs are termed *hyperbolas of order n*.

**6. The exponential and logarithmic functions.** The function $y=a^x$ where $a$ is a constant positive number is called an *exponential function*. The number $a$ is taken positive because for $a<0$ the quantities $a^{\frac{1}{2}}=\sqrt{a}$, $a^{\frac{3}{4}}=\sqrt[4]{a^3}$, etc. would

not be real. The argument $x$ can assume arbitrary real values (Sec. 125). Only positive values of the function $y=a^x$

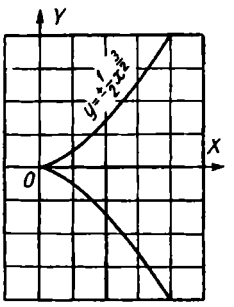


Fig. 244.

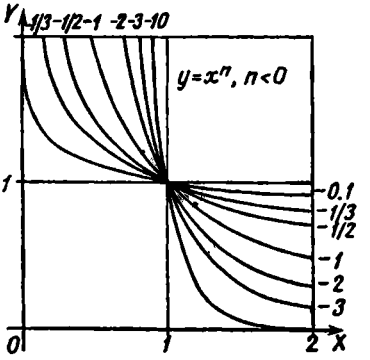


Fig. 245.

are taken. Thus, for the function $y=16^x$, when $x=\frac{1}{4}$, we take only the value $y=2$ and do not consider the value $-2$

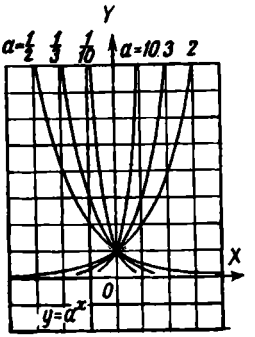


Fig. 246.

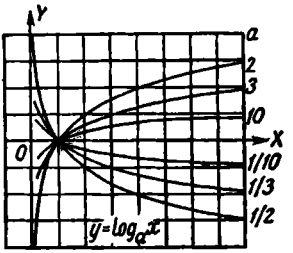


Fig. 247.

(neither, naturally, do we consider the imaginary values $2i$ and $-2i$).

In Fig. 246 are given graphs of the exponential function for $a=\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{10}$, 2, 3, 10. They all pass through the point

(0, 1). (For $a=1$ we have a straight line parallel to the axis of abscissas, the function $a^x$ becomes a constant quantity). For $a > 1$, the graph rises rightwards and for $a < 1$, it falls. All graphs approach without bound the axis of abscissas but never reach it. The graphs of the functions $y=2^x$ and $y=\left(\frac{1}{2}\right)^x$ and also $y=3^x$ and $y=\left(\frac{1}{3}\right)^x$ and, generally, $y=a^x$ and $y=\left(\frac{1}{a}\right)^x$ are symmetric to one another about the axis of ordinates.

The function $y=\log_a x$, where $a$ is a positive constant not equal to 1 (see Secs. 127 and 129, footnote), is called a *logarithmic function.*

A logarithmic function is the inverse of an exponential function. Its graph (Fig. 247) is obtained from the graph of an exponential function (with equal base) by bending the drawing along the bisector of the first quadrant. The graph of any inverse function is obtained in the same manner.

The graph of every logarithmic function is obtained from the graph of every other one by a proportionate change in the ordinate (the logarithms of numbers to different bases are proportional; cf. Sec. 128).

7. **Trigonometric functions. Periodicity.** The definition of a trigonometric function is given in Secs. 182 and 192.

To construct the graph of some trigonometric function (say, the sine) of a variable angle it is necessary to specify on the axis of abscissas a line-segment depicting some definite angle (say 90°) and on the axis of ordinates a segment depicting some number (say 1). We can speak of identical scales on both axes only after it has been established what angle is to be taken as the unit of measure. Only then can the number $x$, which measures the angle, and the number $y$, which yields the sine, be depicted by line segments proportional to the numbers (cf. Sec. 204).

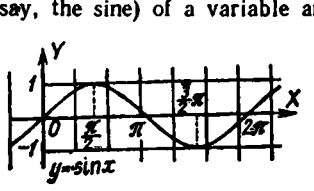


Fig. 248.

The convention in graph construction is to take the radian as the unit of angular measure. Then the function $y=\sin x$ ("$x$ radians" is implied) is represented by the graph in Fig. 248 (the scales on the axes are the same). If for the unit of angular measure we take half a radian, then, retaining the same

scales, we will have to stretch the graph along the axis of abscissas in the ratio 2:1.

The curve (graph) of the function $y = \sin x$ is called a *sine curve (sinusoid)*.

The graph of the function $y = \cos x$ is shown in Fig. 249. This is also a sine curve; it is derived from the graph of $y = \sin x$ by translation along $OX$ by the amount $\frac{\pi}{2}$.

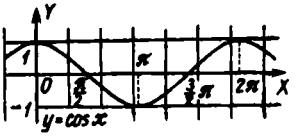


Fig. 249.

Translation (displacement) of the graph of a sine or cosine by the amount $2\pi$ (rightwards or leftwards) brings it to coincidence with itself.

If the graph of some function $y = f(x)$ comes to coincidence with itself upon a translation along the axis of abscissas by some amount, the function is called *periodic*, and the number $p$ which measures the translation is termed the *period* (or *cycle*) of the function $f(x)$. This verbal definition is compactly expressed by the formula

$$f(x + p) = f(x)$$

If $p$ is the period of a function $f(x)$, then $2p$, $3p$, $-2p$, $-3p$ and so forth are also periods.

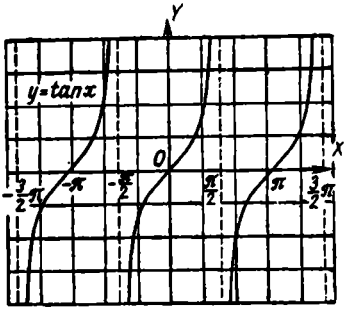


Fig. 250.

All trigonometric functions have a period of $2\pi$.

Besides, the functions $y = \tan x$ and $y = \cot x$ have the period $\pi$ [since $\tan(x \pm k\pi) = \tan x$]. The graph of $y = \tan x$ is given in Fig. 250, the graph of $y = \cot x$ is given in Fig. 251. The tangent curve indefinitely approaches straight lines parallel to the axis of ordinates and distant from it by $\pm\frac{\pi}{2}$, $\pm 3\frac{\pi}{2}$, $\pm 5\frac{\pi}{2}$, etc. (but never reaches these straight lines). For the cotangent curve, a similar role is played by straight lines distant from the $y$-axis ($OY$) by $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, etc. and the $y$-axis ($OY$) itself.

**8. Inverse trigonometric functions.** The definitions of the inverse trigonometric functions were given in Sec. 201 (cf.

Sec. 208). Given here are the graphs of the functions $y = \operatorname{Arcsin} x$ (Fig. 252), $y = \operatorname{Arccos} x$ (Fig. 253), $y = \operatorname{Arctan} x$ (Fig. 254), $y = \operatorname{Arccot} x$ (Fig. 255). They are derived from the graphs of the functions $y = \sin x$, etc., by bending the drawing about

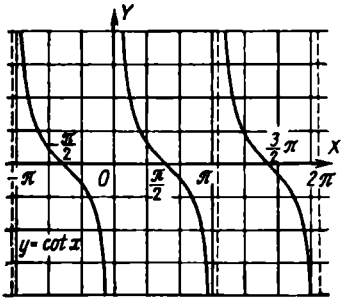


Fig. 251.

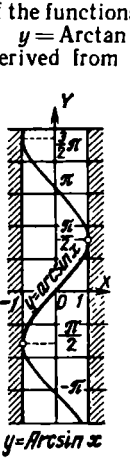


Fig. 252.

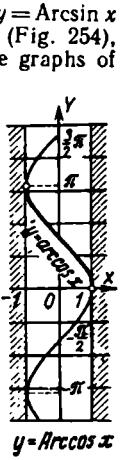


Fig. 253.

the bisector of the first quadrant (cf. Sec. 213, Item 5). The graphs of the functions $y = \operatorname{Arcsin} x$ and $y = \operatorname{Arccos} x$ are located wholly within the vertical strip bounded by the straight

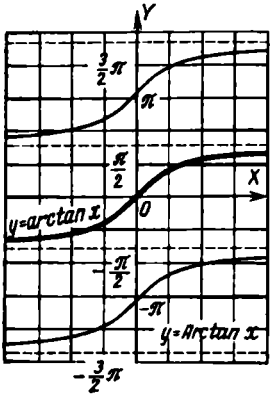


Fig. 254.

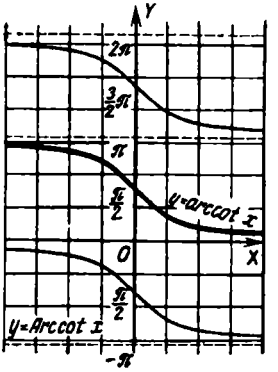


Fig. 255.

lines $x=+1$ and $x=-1$ (these functions do not have real values for $|x|>1$). Each vertical line lying inside the indicated strip intersects the graph an infinite number of times. The same goes for the graphs $y=\operatorname{Arctan} x$ and $y=\operatorname{Arccot} x$, only here the vertical straight line may be taken anywhere. This is due to the multiple-valuedness of the inverse trigonometric functions (Sec. 201). Those portions of the graphs that correspond to the principal values are given as heavy lines in Figs. 252 to 255.

### 214. Graphical Solution of Equations

A graphical representation of functions makes it easy to find an approximate solution to any equation in one unknown or to a system of two equations in two unknowns.

To find the solution of a system of two equations in two unknowns $x$, $y$, we regard each of the equations as a functional relation between the variables $x$ and $y$ and construct two graphs (curves) for these two relations. The coordinates of points common to both curves yield the desired values of the unknowns $x$ and $y$ (the roots of the given system of equations).

**Example 1.** Solve the system of equations

$$\begin{aligned} 7x+5y&=35,\\ -3x+8y&=12 \end{aligned}$$

The graph of each of these equations is a straight line. The line-segments intercepted by the graph of the first equation on the coordinate axes are

$$a=\frac{35}{7}=5, \quad b=\frac{35}{5}=7$$

(Sec. 213, Item 2). Using these segments, construct the straight line $AB$ (Fig. 256). In the same way, we find $a=-4$, $b=1.5$ for the graph of the second equation and construct the straight line $CD$.*

The coordinates of the point $K$ of intersection of the graphs yield the required values of $x$ and $y$. We take the values of the coordinates by inspection: $x(=OP)=3.1$, $y(=PK)=2.7$. The exact values of the roots are $x=3\frac{7}{71}$, $y=2\frac{47}{71}$.

---

* Instead of finding the segments $a$ and $b$, you can plot any two points of the straight line. To do this, assign any two values to $x$ and compute the corresponding values of $y$.

**Example 2.** Solve the equation $\frac{1}{2}x^2-\frac{1}{2}x-2=0$. It can be solved graphically as an equation in one unknown (see Example 4 below), but it is easier to replace it by a system of equations:

$$y=\frac{1}{2}x^2,\quad y=\frac{1}{2}x+2$$

and then solve the system graphically.

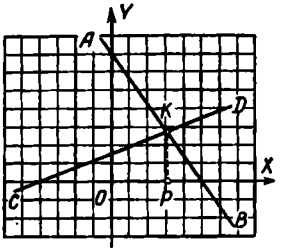


Fig. 256.

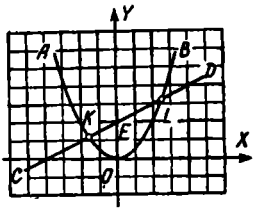


Fig. 257.

The first equation is graphically depicted in Fig. 257 as a parabola $AOB$ (Sec. 213, Item 4) which can be plotted by points. The graph of the second equation is a straight line $CD$ which intercepts on the axis of ordinates a line-segment $b(=OE)=2$; the slope is $m(=\tan<DCX)=\frac{1}{2}$ (Sec. 213, Item 2). We find two points $K$ and $L$ at the intersection of the straight line $CD$ with the parabola $AOB$. The abscissas of these two points (found by inspection) $x_1=-1.6$ and $x_2=2.6$ yield the approximate values of the roots of the given equation. The exact values of the roots are

$$x_1=\frac{1-\sqrt{17}}{2},\quad x_2=\frac{1+\sqrt{17}}{2}$$

**Example 3.** Solve the equation $2^x=4x$. This equation cannot be reduced to an algebraic equation. One of the roots ($x=4$) is easily found. To find the other roots (if they exist), it is best to begin with a graphical solution. Replace the given equation by a system: $y=2^x$, $y=4x$. Plot (Fig. 258) the graph of the exponential function $y=2^x$ (by points, assigning to the argument the values $x=-1, 0, 1, 2, 3$, etc.)

and of the function $y=4x$ (a straight line). Here the ordinates grow much faster than the abscissas; it is therefore better to choose a smaller scale unit for the $x$-axis ($OX$) than for the $y$-axis ($OY$). In Fig. 258 the difference is a factor of four.

At the intersection we find two points, $A$ and $B$. We can see from the construction that the curves have no other common points. The abscissa of $A$ is $x=4$. The abscissa of $B$ is (by eye) about $x \approx 0.3$.

The solution thus found can be made precise by computation. Using

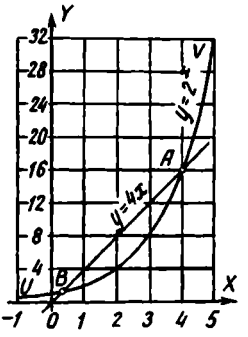


Fig. 258.

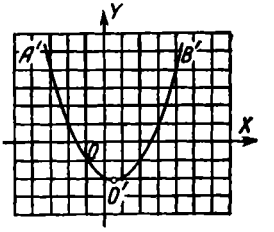


Fig. 259.

tables of logarithms, we find the value of $2^x$ for $x=0.3$. It is 1. 231. This number is somewhat greater than $4x=1.200$ (by 0.031). Hence (see the graph) the number 0.3 is less than the abscissa of point $B$. Let us test the value $x=0.35$. It yields $2^x=1.275$; $4x=1.400$. Now $2^x$ is less than $4x$ (by 0.125). Hence the number 0.35 is greater than the abscissa of $B$ so that the true value of $x$ lies between 0.30 and 0.35, being roughly 4 times closer to the former value than to the latter (since 0.031 is 4 times smaller than 0.125). And so $x \approx 0.31$. A check yields $2^x=1.240$, $4x=1.240$. Incidentally, $x=0.31$ is not the exact root. If we take a higher-place logarithmic table there will be a difference in the fifth significant digit between $2^x$ and $4x$. In the same way we can find a more precise value for the root.

To find the solution of an equation in one unknown, it is possible, by transposing all terms to the left member, to represent it as $f(x)=0$. Construct the graph of the function $y=f(x)$. The abscissas of the points of intersection of this graph with the axis of abscissas will be the roots of the given equation.

**Example 4.** Solve the equation $\frac{1}{2}x^2 = \frac{1}{2}x + 2$. Transpose all terms to the left member: $\frac{1}{2}x^2 - \frac{1}{2}x - 2 = 0$. Plot the graph of the function $y = \frac{1}{2}x^2 - \frac{1}{2}x - 2$ (by points). We get (Fig. 259) the parabola $A'O'B'$. Its shape is the same as in the preceding example; the vertex lies in point $O'\left(\frac{1}{2}, -2\frac{1}{8}\right)$ (see Sec. 213, Item 4). We find two points at the intersection of the graph with the axis of abscissas. Reading off the abscissas, we get $x_1 = -1.6$, $x_2 = 2.6$.

### 215. Graphical Solution of Inequalities

A graphical solution of an inequality (like that of an equation) is not very accurate. However, the graphical method is pictorial and readily surveyable, and in the solution of inequalities (particularly systems of inequalities) these features are still more valuable than in the solution of equations. The methods of solution are the same as for equations (Sec. 214); the solutions however are depicted as line-segments, not points.

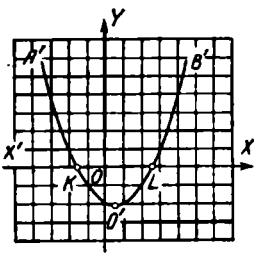

Fig 260

**Example 1.** Solve the inequality

$$\frac{1}{2}x^2 - \frac{1}{2}x - 2 < 0$$

Construct (Fig. 260) the graph of the function $y = \frac{1}{2}x^2 - \frac{1}{2}x - 2$ (cf. Sec. 214, Example 4). By hypothesis, $y < 0$; hence, the points corresponding to the solution must lie below the axis of abscissas. The graph shows that the locus of these points is an arc, $KO'L$, of the parabola $A'O'B'$ (the extremities $K$ and $L$ of the arc are excluded; for them, $y = 0$). To the values of $x$ which satisfy the given inequality there correspond interior points of the line-segment $KL$ of the axis of abscissas. We read off the graph $-1.6 < x < 2.6$. If an exact solution is required, find the abscissas of the points $K$ and $L$ computationally; that is, solve the quadratic

equation

$\frac{1}{2}x^2-\frac{1}{2}x-2=0$. Then we find $\frac{1-\sqrt{17}}{2}<x<\frac{1+\sqrt{17}}{2}$.

**Example 2.** Solve the inequality

$$\frac{1}{2}x^2-\frac{1}{2}x-2>0.$$

Construct the same graph as in Example 1. This time we must have $y>0$; that is the points must lie above the axis of abscissas. The locus of these points are the lines $KA'$ and $LB'$ which extend upwards without bound (the starting points, $K$ and $L$, are excluded). The appropriate points of the axis of abscissas fill the rays $KX'$ and $LX$ (points $K$ and $L$ excluded). This inequality holds true: (1) for $x<-1.6$, and (2) for $x>2.6$. The exact solution is

$$(1)\ x<\frac{1-\sqrt{17}}{2},\quad (2)\ x>\frac{1+\sqrt{17}}{2}$$

**Example 3.** Solve the inequality

$$\frac{1}{2}x^2<\frac{1}{2}x+2$$

This inequality is equivalent to the inequality $\frac{1}{2}x^2-\frac{1}{2}x-2<0$ solved in Example 1, but in the form given here it is easier to solve.

Construct (cf. Sec. 214, Example 2) the graphs of the functions $y=\frac{1}{2}x^2$ (parabola $AOB$ in Fig. 261) and $\bar{y}=\frac{1}{2}x+2$ (straight line $CD$). The bar on $y$ is used to distinguish the ordinate of the straight line from the ordinate of the parabola for the same abscissa. By hypothesis, we must have $y<\bar{y}$, that is, the points of the parabola must lie below the points of the straight line with the same abscissas. The graph shows that the corresponding pieces of the lines $AOB$ and $CD$ (arc $KOL$ and line-segment $KL$) lie above $K_1L_1$ (heavy line) of the axis of abscissas (extremities $K_1$ and $L_1$ are excluded). Reading off the abscissas of the points $K$ and $L$, we find the (approximate) solution $-1.6<x<2.6$.

**Example 4.** Solve the inequality $\frac{1}{2}x^2<x-3$.

Construct (Fig. 262) the graphs of the functions $y=\frac{1}{2}x^2$ (parabola $AOB$) and $\bar{y}=x-3$ (straight line $CD$). We must

have $y < \bar{y}$. But the parabola $AOB$ lies entirely above the straight line $CD$. This inequality does not have a solution.

**Example 5.** Solve the inequality $\frac{1}{2}x^2 > x-3$. The construction is the same as in the preceding example. But here, $y > \bar{y}$; therefore the given inequality is unconditional.

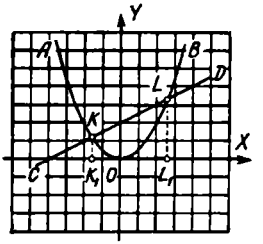


Fig. 261.

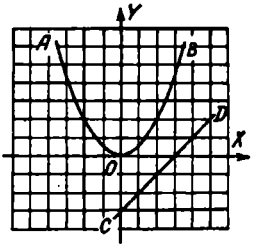


Fig. 262.

**Example 6.** Solve the system of inequalities:

$$x+4 \leqslant x^2 \leqslant 6-x, \quad \frac{1}{2}x^2 > \frac{3}{2}-\frac{1}{4}x$$

In place of the first two inequalities we can write the following equivalent ones: $\frac{1}{2}x+2 \leqslant \frac{1}{2}x^2 \leqslant 3-\frac{1}{2}x$. Construct (Fig. 263) the graphs of the functions $y=\frac{1}{2}x^2$ (parabola $AOB$); $y'=\frac{1}{2}x+2$ (straight line $CD$); $y''=3-\frac{1}{2}x$ (straight line $UV$); $\bar{y}=\frac{3}{2}-\frac{1}{4}x$ (straight line $EF$). The first two inequalities require that the arc of the parabola lie above the straight line $CD$ and below the straight line $UV$ or have common points with these lines. We thus isolate arc $RP$ (extremities $R$, $P$ included) on the parabola, and line-segment $R_1P_1$ on the axis of abscissas. The third inequality requires that the arc of the parabola also pass higher than the straight line $EF$. In

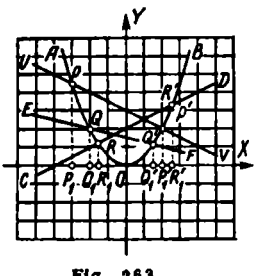


Fig. 263.

this way, we isolate arc $QP$ (extremity $P$ included and extremity $Q$ excluded) from the arc $RP$, and on the axis of abscissas we find $P_1Q_1$. Reading off the abscissas of points $Q$ and $P$, we have $-3 \leqslant x < -2$.

**Example 7.** Solve the inequality $\frac{x^2+x-6}{x^2-x-4} < 0$.

This inequality is valid in two cases:

(1) when $x^2+x-6 < 0$ and at the same time $x^2-x-4 > 0$,

(2) when $x^2+x-6 > 0$ and at the same time $x^2-x-4 < 0$.

In the former case we have $x+4 < x^2 < 6-x$. The solution of this system (see Example 6) is graphically given by the line-segment $P_1R_1$ (extremities $P_1$ and $R_1$ excluded). In the latter case we have $x+4 > x^2 > 6-x$. Solving this system in the same way as the preceding one, we find the arc $P'R'$ of parabola $AOB$ and the corresponding line-segment $P_1'R_1'$ of the axis of abscissas (extremities $P_1'$ and $R_1'$ excluded). Reading off the abscissas of points $P$, $R$, $P'$, $R'$, we find that the given inequality is satisfied (1) for $-3 < x < -1.6$ and (2) for $2 < x < 2.6$.

**Example 8.** Solve the inequality $2^x < 4x$.

Construct the graphs of the function $y=2^x$ (curve $UV$ in Fig. 258, page 395) and of the function $\bar{y}=4x$ (straight line $AB$). By hypothesis, $y < \bar{y}$, that is the points of the curve $UV$ must be below the points of the straight line $AB$. Reading off the abscissas of the points $A$ and $B$, we find the solution: $0.3 < x < 4$.

### 216. Analytical Geometry (Fundamental Notions)

In elementary geometry, the solution of every separate problem requires a certain amount of ingenuity and it often happens that problems which are extremely similar require quite different techniques of solution that are not always easy to hit upon. For instance, take the problem: to find the locus of points $M$ whose distances $MA$ from a point $A$ are equal to the distances $MB$ from a given point $B$. As we know, the desired locus is a straight line (perpendicular to the midpoint of $AB$). Now the method ordinarily used in elementary geometry to solve this problem is not suitable for the following problem: find the locus of points $M$ whose distance $MA$ from point $A$ is twice the distance $MB$ from point $B$.

Analytical geometry, which was constructed at the same time by two French scientists—Descartes (1596—1650) and Fermat (1601—1655)—gives *uniform techniques for solving*

*geometrical problems* and reduces the solution of a broad range of problems to a few regularly used methods. This is done in the following manner: all the given and required points and lines are *referred to a system of coordinates* (it is immaterial what system is chosen but one often finds that an apt choice simplifies the solution of a problem). Having chosen a system of coordinates, we can describe each point by its coordinates and each line by an equation whose graph this line is. A given geometric problem is thus reduced to an algebraic problem, and we have well-elaborated general methods in our possession to solve algebraic problems.

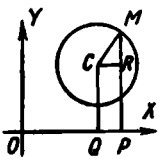

Fig. 264.

This can be illustrated by the following examples.

**Example 1.** A circle of radius $r$ is referred to the coordinate system $XOY$ (Fig. 264) in which the centre $C$ has abscissa $OQ=a$ and ordinate $QC=b$. Set up the equation of the circle.

Let $M(x, y)$ be an arbitrary point on the circle ($x=OP$, $y=PM$). The length of $MC$ is always equal to the constant $r$, by the definition of a circle. Express $MC$ in terms of the constant coordinates $a$ and $b$ of centre $C$ and the variable coordinates $x$ and $y$ of point $M$. From Fig. 264 we have

$$MC=\sqrt{CR^2+RM^2}=\sqrt{(OP-OQ)^2+(PM-QC)^2}$$
$$=\sqrt{(x-a)^2+(y-b)^2}$$

Consequently

$$\sqrt{(x-a)^2+(y-b)^2}=r$$

or

$$(x-a)^2+(y-b)^2=r^2 \qquad (1)$$

This equation represents a circle; in other words, the graph of equation (1) is a circle.

**Example 2.** Find the locus of points $M$ for which $MA=2MB$ ($A$ and $B$ are two given points separated by a distance of $2l$).

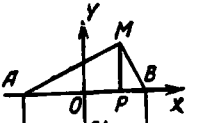

Fig. 265.

Take the origin of coordinates at the midpoint $O$ of $AB$ and draw one of the axes ($OX$ in Fig. 265) along $AB$. In order to write the condition $MA=2MB$ as an equation between the coordinates of point $M(x, y)$, express $MA$ and $MB$ in terms of the coordinates. From the triangle $MBP$ we have

$MB = \sqrt{PB^2 + PM^2} = \sqrt{(OB - OP)^2 + PM^2} = \sqrt{(l-x)^2 + y^2}$
In the same way, from triangle $AMP$ we find $MA = \sqrt{(x+l)^2 + y^2}$ and the condition $MA = 2MB$ takes the form

$$\sqrt{(x+l)^2 + y^2} = 2\sqrt{(l-x)^2 + y^2}$$

Simplifying we have

$$x^2 - \frac{10}{3} lx + y^2 + l^2 = 0 \qquad (2)$$

The desired locus is the graph of this equation, and the methods of analytical geometry permit stating at once that this graph is a circle. This is evident if we compare (2) and (1). Changing equation (2) to the form

$$\left(x - \frac{5}{3} l\right)^2 + y^2 = \left(\frac{4}{3} l\right)^2$$

we see that it is a special case of equation (1) when $a = \frac{5}{3} l$, $b = 0$, and $r = \frac{4}{3} l$. Hence our locus is a circle with centre at point $C\left(\frac{5}{3} l, 0\right)$ and radius $r = \frac{4}{3} l$.

### 217. Limits

A constant $a$ is the limit of a variable $x$ if the variable approaches $a$ without bound. *

It is essential to bear in mind that when considering a separate variable one cannot speak of finding its limit. But if we consider two variables and one is a function of the other, then for one of them (argument) it is possible to specify the limit, and for the other, to seek the limit (if it exists).

**Example 1.** The variables $x$ and $y$ are connected by the relationship $y = \frac{x^2 - 4}{x - 2}$; find the limit of $y$ when $x$ has 6 as its limit.

*The definition given here is not quite rigorous enough since the expression "approaches without bound" must be logically refined. It is hardly possible to make it more precise briefly and in appropriate fashion. The examples that follow will elucidate the meaning of limit to the extent that is necessary here. The definitions frequently found in elementary textbooks suffer from the same lack of comprehensiveness, though outwardly they often appear to be more precise.

Let the variable $x$ approach the number 6 without bound in some way; for instance, we assign to $x$ the values 6.1, 6.01, 6.001 and so on. We find the values of $y$: 8.1, 8.01, 8.001, etc. These values approach the number 8 without bound. The same will occur if we let $x$ approach 6 without bound *in any other way*; say, putting $x=5.9$, 6.01, 5.999, 6.0001, etc. For this reason, when $x$ has the limit 6, $y$ has the limit 8. We write this as follows:

$$\lim_{x \to 6} y = 8 \quad \text{or} \quad \lim_{x \to 6} \frac{x^2-4}{x-2} = 8$$

In the given case we could have obtained this result by putting $x=6$ in the expression $y=\frac{x^2-4}{x-2}$. In the example which follows this method would not succeed.

**Example 2.** Given $y=\frac{x^2-4}{x-2}$. Find $\lim_{x \to 2} y$. Substituting $x=2$ into $\frac{x^2-4}{x-2}$ we get the indeterminate expression $\frac{0}{0}$ (Sec. 37). Yet computations like those performed in Example 1 show that $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$. This result could also have been obtained thus: we have $\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}$. When $x \neq 2$, we can cancel out $x-2$ (cancellation is not legitimate for $x=2$). We get $y=x+2$ (when $x \neq 2$). Let $x$ approach 2 without bound though *never reaching* the value $x=2$; then $y$, remaining equal to $x+2$, approaches 4 without bound.

This problem is sometimes formulated thus: "find the true value of the expression $\frac{x^2-4}{x-2}$ for $x=2$" or "evaluate the indeterminate expression $\frac{x^2-4}{x-2}$ for $x=2$". The precise meaning of these expressions is to find the limit: $\lim_{x \to 2} \frac{x^2-4}{x-2}$.

In the example at hand, evaluating the indeterminate expression is attained by cancelling $x-2$ out of the fraction $\frac{x^2-4}{x-2}$ with the subsequent substitution $x=2$. But neither does this device always lead to the proper result.

### 218. Infinitely Small and Infinitely Large Quantities

A variable whose limit is zero is termed an *infinitely small quantity (infinitesimal)*.

**Example 1.** The variable $\sqrt{x+3}-2$ is an infinitesimal if $x$ tends to 1 because

$$\lim_{x \to 1} (\sqrt{x+3}-2)=0$$

A variable that constantly increases in absolute magnitude is termed an *infinitely large quantity.*

**Example 2.** The variable $\frac{x}{x-5}$ is an infinitely large quantity if $x$ tends to 5.

Infinitely large quantities do not have any limits. Nonetheless, it is conventional to say that an infinitely large quantity "tends to an infinite limit". Accordingly we write

$$\lim_{x \to 5} \frac{x}{x-5}=\infty \qquad (1)$$

The symbol $\infty$ (infinity) does not denote any number, and (1) is not a real equation but simply expresses the fact that, as $x$ approaches 5 without bound, the absolute value of the fraction $\frac{x}{x-5}$ increases without bound. Here, the fraction can assume either positive values (when $x > 5$) or negative values (when $x < 5$).

**Note.** In other cases, an infinitely large quantity may assume only positive (or only negative) values. Thus, if $x$ is an infinitesimal, the quantity $\frac{1}{x^2}$ is infinitely great; but both when $x > 0$ and when $x < 0$ the quantity $\frac{1}{x^2}$ is positive. In symbols this is expressed as: $\lim_{x \to 0} \frac{1}{x^2}=+\infty$. Contrariwise, the quantity $-\frac{1}{x^2}$ is always negative, and we write $\lim_{x \to 0} \left(-\frac{1}{x^2}\right)=-\infty$.

Accordingly, the result of Example 2 can also be written thus: $\lim_{x \to 5} \frac{x}{x-5}=\pm\infty$.

**Example 3.** The notation $\lim_{x \to \infty} \frac{x-1}{x}=1$ means that when $x$ is infinitely great (i. e., when $x$ increases without bound in absolute value), the quantity $\frac{x-1}{x}$ tends to the limit 1.

The symbol $x \longrightarrow \infty$ is read: "$x$ approaches (or tends to) infinity".

**Example 4.** The expression "the area of a circle is the limit of the area of a regular inscribed polygon when the number of sides is infinite" means that, as the number of sides of the polygon increases without bound, the area of the polygon approaches without limit the area of the circle.

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